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VIII. *Exhibition and Description of some Apparatus for Class Work in Practical Physics. By Dr.* G. F. C. SEARLE, *F.R.S., University Lecturer in Experimental Physics, Cambridge.*

DR. SEARLE exhibited and described the apparatus used for the following experiments in his class at the Cavendish Laboratory, Cambridge. The methods are briefly described in the following abstracts ; the references indicate where fuller information may be found. In a few cases the complete Papers have not been published at the date of publication of these abstracts, but it is hoped that they will be published during the year 1915. The titles of these Papers are distinguished by an asterisk (*).

1. *An Experiment on the Harmonic Motion of a Rigid Body.**

The experiment is designed to illustrate the formula

$$T=2\pi\sqrt{K/\mu},$$

where T seconds is the periodic time of a rigid body of moment of inertia K grm.cm.^{2} which is suspended from a fixed support by a torsion wire exerting a restoring couple of μ dyne-cm. per radian. When μ is to be found, a vertical cylinder is attached to the lower end of the wire and two threads are wound round the cylinder. These threads pass over ball-bearing pulleys and support known masses. The angular deflexion of the cylinder caused by the masses is measured with a goniometer by the method of auto-collimation. An inertia bar of known moment of inertia K is then substituted for the cylinder and the periodic time is observed. ("Proc." Cambridge Phil. Soc., Vol. XVIII., p. 31 ; also "Experimental Harmonic Motion," Cambridge University Press, 1915. This small manual is similar in character to the author's "Experimental Elasticity.")

2. *Determination of Gravity by a Rigid Pendulum.**

The periodic time, T, of a rigid pendulum vibrating about a horizontal axis (knife edge) is given by

$$T=2\pi\sqrt{K/Mgh}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where K is the moment of inertia of the pendulum about its

axis of suspension, h is the distance of the centre of gravity from that axis and M is the mass of the pendulum. Since K cannot be satisfactorily calculated from the mass and dimensions of the pendulum, K must be found in some other way. By the theorem of parallel axes,

$$K = K_0 + Mh^2, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where K_0 is the moment of inertia of the pendulum about an axis through its centre of gravity parallel to the knife edge.

To determine K_0, the pendulum is suspended from a torsion wire by aid of a simple stirrup so that the rod of the pendulum is horizontal. The periodic time, T_0, of the torsional vibrations is found. An inertia bar of calculable moment of inertia K_1 is then attached to the torsion wire in place of the stirrup and pendulum and the periodic time, T_1, is observed. Then

$$K_0 = K_1 T_0^2 / T_1^2.$$

This value of K_0 is used in (2) and so K is found. The distance h is found by balancing the pendulum on a fulcrum. (Searle, "Experimental Harmonic Motion," Cambridge University Press, 1915.)

3. *Determination of Poisson's Ratio for Indiarubber.**

Let the length of a tube of circular section be l, let its internal radius be r, and let its internal volume be v, so that $v = \pi l r^2$. Let the length be slightly increased so that l becomes l', r becomes r' and v becomes v'. Then, if σ is Poisson's ratio,

$$\sigma = \tfrac{1}{2}\left\{1 - \frac{l}{v} \cdot \frac{v' - v}{l' - l}\right\}.$$

For infinitesimal changes we have

$$\sigma = \tfrac{1}{2}\left\{1 - \frac{l}{v} \cdot \frac{dv}{dl}\right\}.$$

An indiarubber tube is used. The lower end is closed by a solid plug; in the upper end a glass tube of small bore is inserted. The distance between the end of the plug and that of the glass tube is taken as l. The indiarubber and glass tubes contain water, and the change in the position of the meniscus when the rubber tube is stretched is observed. From the resulting curve the value of dv/dl is obtained. Proper clamps are provided for securing the rubber tube to the plug and to the glass tube, and other fittings allow these clamps to be held in position on a vertical steel rod. Scales for reading l and the

position of the meniscus are provided. The value of σ is about 0·47. (Searle, "Experimental Elasticity," second edition, 1915.)

4. *Two Methods of Measuring the Surface Tension of Spherical Soap Films.*

In one method the pressure excess due to a curved soap film is measured by aid of what may be called a "viscosity potentiometer." Air from a gasometer or a gas-bag flows through two tubes, AB, BC, in series. The pressure at A is measured by a manometer; the end C is open to the air. From the junction B a side tube leads to a cup with a horizontal circular rim on which a soap film is placed. On account of the viscosity of the air, there is a fall of pressure along each flow tube. For a given flow of air, the fall of pressure in either tube is proportional to the length of the tube and inversely proportional to the fourth power of its internal radius. The excess of the pressure at B over that of the atmosphere causes the film to become part of a sphere. From the distance of the highest point of the film above the plane of the rim and from the radius of the rim, the radius, r, of the film can be computed. If the length of the tube AB be l_1, and its internal radius be a_1, and if l_2 and a_2 be the length and radius of BC, and if the pressure excess at B be p_B and that at A be p_A; then

$$\frac{p_A - p_B}{p_B} = \frac{l_1 a_2^4}{l_2 a_1^4}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

from which p_B can be found; p_B in the experiment shown corresponded to about 0·03 cm. of water. The surface tension, T, of the film is then found from $4T = rp_B$. It is assumed in (1) that the variation of density of the air along ABC is negligible.

In the second method the pressure excess is measured by a device employed by Mr. J. D. Fry in the calibration of his micro-manometer ("Phil. Mag.," April, 1913, p. 494). The method depends upon the difference of density between cold and hot air at the same pressure. A metal tube, ABCD, has the two parts, AB, CD, each a few centimetres long, at right angles to the main portion BC, which is about 1 metre in length. This bent tube is surrounded by a second tube, which is used as a steam jacket for heating the tube ABCD. The parts AB, CD are horizontal and the tube may, if desired, be rotated about CD as a horizontal axis. The inner tube passes out of the steam jacket through rubber bungs. The opening D is connected by

a horizontal rubber tube to a cup with a cylindrical rim on which a film is formed; the lower end A is open to the atmosphere. If A is at a depth z below D, the pressure excess, p, within the bubble is given by

$$p = gz(\rho_1 - \rho_2),$$

where the densities ρ_1 and ρ_2 of the cold and hot air are given by

$$\rho_1 = \rho_0 P t_0 / p_0 t_1, \quad \rho_2 = \rho_0 P t_0 / p_0 t_2.$$

Here ρ_0 is the density of air at normal pressure p_0 and normal absolute temperature t_0^0, P is the atmospheric pressure at the time of the experiment, t_1 is the atmospheric temperature near the tube and t_2 is the temperature of the steam. The radius, r, of the bubble is measured and the surface tension is calculated by $T = \frac{1}{4}rp$. ("Proc.," Cambridge Phil. Soc., Vol. XVII., p. 285.)

5. *A Simple Viscometer for Very Viscous Liquids.*

If the space between two coaxial cylinders of radii a, b, and of length h be filled with viscous liquid, the viscosity μ is given in terms of the couple G, which maintains the inner cylinder in motion about its axis with angular velocity ω relative to the outer fixed cylinder, by the equation

$$\mu = \frac{G(a^2 - b^2)}{4\pi\omega h a^2 b^2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

The apparatus exhibited to the Society is adapted for finding the viscosity of treacle. The axes of the cylinders are vertical and the outer cylinder is fixed. The inner cylinder is carried by an axle furnished at each end with a pivot working in a fixed bearing. Two threads wound round a drum fixed to the axle pass over ball-bearing pulleys and support weights which drive the inner cylinder round at a slow speed when the space between the cylinders contains treacle. The angular velocity is found to be proportional to the driving couple.

Means are provided for varying the length of the inner cylinder which is exposed to the action of the treacle and in this way a correction is found for the lower end, where the motion of the liquid differs from that contemplated in the theory leading to (1). The arrangement is self-contained so as to avoid the changes of temperature which would result if fresh treacle were added to the system. When the correction has been found, the value of μ can be found from the equation.

At 12°C. the viscosity of treacle is about 400 in C.G.S. units, and that of "golden syrup" is about 1,000; the viscosity of water at the same temperature is 0·0146. ("Proc.," Cambridge Phil. Soc., Vol. XVI., p. 600.)

6. *Experiments with a Prism of Small Angle.*

When a ray in a principal plane passes nearly symmetrically through a prism of small angle i radians placed in air, the deviation, D_1, is nearly independent of the angle of incidence and has the approximate value

$$D_1=(\mu-1)i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where μ is the refractive index of the prism.

If the prism is placed symmetrically in a tank with parallel glass ends, the deviation is

$$D_2=(\mu-\mu_2)i, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

where μ_2 is the refractive index of the liquid contained in the tank.

The cross-wire of a collimator set for "infinity" is illuminated by sodium light. The direction of the emergent rays corresponding to the cross-wire is observed by a goniometer, the image of the collimator wire being focussed on the cross-wire of the goniometer.

The difference of deviation observed when the prism is turned in air from one position of minimum deviation to the other gives $2D_1$, and the corresponding difference when the prism is immersed in the liquid gives $2D_2$. If a liquid of known index, such as water, is used, (1) and (2) give the angle of the prism and its refractive index. When these have been found, the refractive index μ_3 of any other liquid can be found from the deviation D_3 observed when the tank contains that liquid.

Thus $$D_3=(\mu-\mu_3)i \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

The prisms supplied by opticians for use in spectacles are suitable for this experiment. ("Proc.," Cambridge Phil. Soc., 1915.)

7. *Revolving Table Method of Determining the Curvature of Spherical Surfaces.*

In this method, designed by the author in conjunction with Mr. A. C. W. Aldis and Mr. G. M. B. Dobson, the radius of a

spherical reflecting surface is found directly from two readings on a straight uniformly divided scale, without corrections or calculations of any sort.

A table turning about a vertical axis is required ; the plane of the top of the table is normal to the axis of revolution, and the top carries a straight scale, against which slides a carriage bearing the spherical surface. When the apparatus is in adjustment, the straight line described by the centre of curvature of the spherical surface, when the carriage slides along the scale, *intersects* the axis of revolution of the table. The position of the carriage relative to the table top, when the centre of curvature lies on the axis of revolution of the table, will be called the *first position.* If the table be turned through any angle about the vertical axis, when the carriage is in the first position, the only effect of the motion is to substitute one part of the spherical surface for another. Hence, if rays from an object fall upon the surface, the reflected rays will be unaffected by the motion. This furnishes a means of setting the surface in the first position.

The carriage is now moved into a *second position*, in which the vertical axis of the table is a tangent line to the spherical surface. If, now, the table be turned about its axis, a grain of lycopodium placed on the surface at the point of contact of the vertical tangent line will remain stationary.

The radius of curvature of the surface is given by the difference of the two scale readings of the carriage in the first and second positions.

The adjustments of the surface to be tested are facilitated by the use of a small lathe head to form the " carriage." (" Phil. Mag.," February, 1911, pp. 218-224. Also " Proceedings " of the Optical Convention, Vol. II., p. 161, 1912.)

8. *Experiments Illustrating Flare Spots in Photography.*

When light from a point S falls on a simple thin lens of focal length f, most of it passes through the lens, and forms an image of S. But some of the light suffers two reflections within the lens, and this light gives rise to a second image of S of small intensity, the corresponding focal length being $(\mu-1)f/(3\mu-1)$, where μ is the refractive index. This image is called a " flare spot." When two lenses are used there are six flare spot images of any object formed by twice reflected rays, and with t lenses there are $t(2t-1)$ such images.

In the case of two thin lenses AB (radii a, b) and CD (radii

c, d) placed with the faces B, C in contact, the "powers" for the six flare spots are as follows, where the subscript letters indicate the surfaces at which reflection has occurred :—

$$P_{DC}=M+N_2, \qquad P_{DB}=M+N_2+W,$$
$$P_{DA}=M_2+N_2+W, \qquad P_{CB}=M+N+W,$$
$$P_{CA}=M_2+N+W, \qquad P_{BA}=M_2+N.$$

Here M is the "power" $=$(focal length)$^{-1}$ of AB, and M_2 is the power of AB for the flare spot when AB is used alone, and N and N_2 are the corresponding powers of CD. Also $W=-2(1/b+1/c)$. Each radius is counted positive when the corresponding surface is convex. Following the rule of the practical opticians, the focal length of a converging lens is counted positive.

If each of the two lenses is a converging meniscus, M, M_2, N, N_2 are positive, and W is also positive if the concave surfaces face each other. In this case the system has a positive power for each of the six flare spots, and thus six real flare spot images of a real object can be formed. "Periscopic" spectacle lenses are convenient for the experiment. The values of the five quantities M, M_2, N, N_2, W are found by experiment, and the six powers P_{DC}, &c., are calculated from them. The values are found to be in good agreement with the six powers found when the complete system of two lenses is used.

If a suitable converging lens is placed between the two meniscus lenses, the 15 secondary images are easily seen. ("Proc.," Cambridge Phil. Soc., Vol. XVII., p. 205.)

9. *Determination of the Effective Aperture of a Photographic Lens.**

The stops of a lens are marked with such symbols as $f/8$. The symbol $f/8$ means that the effective diameter of the stop is one-eighth of the focal length of the lens system. The effective diameter of the stop is not the diameter of the hole in the diaphragm, but is the diameter of that incident beam of rays parallel to the axis which in its passage through the system exactly fills the opening in the actual diaphragm.

When the stop is placed in front of the lens system, its effective diameter is simply equal the actual diameter of the stop itself.

In most cases the stop S is placed between the components of

the lens system. Let T be the image of S formed by the front lens L. Then S is also the image of T, and a ray, RQ, which before incidence on L is parallel to the axis, and is directed to a point, Q, on the edge of T, will, after passing through L, pass through P, the corresponding point on the edge of S. Hence, the distance of RQ from the axis, and, therefore, the radius of T, is equal to the effective radius of the stop.

The effective diameter may be measured by a microscope mounted on a sliding carriage. The axis of the microscope is parallel to the axis of the lens system, and the carriage moves at right angles to that axis. The microscope is focussed through the lens L upon the stop, and the diameter of the image T is measured by aid of the sliding carriage.

If a point source of light is placed at that principal focus, F, of the system which lies on the photographic plate when distant objects are in focus, the rays from this source will, after passing through the system, form a parallel beam. If this beam is received on a translucent screen, A, placed in front of L, the diameter of the luminous patch is the effective diameter of the stop.

Since an infinitely small source cannot be obtained, a correction becomes necessary.

In practice, a small hole of diameter h in a metal plate in the focal plane is illuminated by a flame to serve as the source of light. If the diameter of the bright patch on the translucent screen is c, then the effective diameter a of the stop is given by

$$a=c-xh/f,$$

where f is the focal length of the complete lens system, and x is the distance of the image T from the matt surface of the screen. The distance x is measured by aid of the microscope and sliding carriage, the axis of the microscope being now parallel to the direction of motion of the carriage.

The determination of x may be avoided if two holes of diameter h_1, h_2 are used, the corresponding diameters of the bright patch being c_1, c_2. Then

$$\frac{c_1-a}{c_2-a}=\frac{h_1}{h_2},$$

whence $$a=c_1-\frac{h_1(c_1-c_2)}{h_1-h_2}.$$

("Proc.," Cambridge Phil. Soc., 1915.)

10. *The Comparison of Nearly Equal Resistances.*

Four resistance coils A, B, C, D, are arranged to form the four sides of a Wheatstone's quadrilateral. The coils C, D are approximately equal, but, as their ratio is eliminated, it is not necessary to know it. A balance is obtained by shunting A, B with large resistances a_1, b_1*. The coils A and B are then interchanged, and a fresh balance is obtained by shunting them with a_2 and b_2. Then

$$C\left(\frac{1}{A}+\frac{1}{a_1}\right)=D\left(\frac{1}{B}+\frac{1}{b_1}\right)$$

and

$$D\left(\frac{1}{A}+\frac{1}{a_2}\right)=C\left(\frac{1}{B}+\frac{1}{b_2}\right).$$

Hence

$$\left\{\left(\frac{1}{A}+\frac{1}{a_1}\right)\left(\frac{1}{A}+\frac{1}{a_2}\right)\right\}^{\frac{1}{2}}=\left\{\left(\frac{1}{B}+\frac{1}{b_1}\right)\left(\frac{1}{B}+\frac{1}{b_2}\right)\right\}^{\frac{1}{2}}.$$

In practice, A is so nearly equal to B and C is so nearly equal to D that a_1, a_2, b_1, b_2 are very large compared with A and B, and the arithmetic means may be used instead of the above geometrical means. Then

$$\frac{1}{A}+\frac{1}{2}\left(\frac{1}{a_1}+\frac{1}{a_2}\right)=\frac{1}{B}+\frac{1}{2}\left(\frac{1}{b_1}+\frac{1}{b_2}\right),$$

which gives B in terms of A when the shunts are known. The details of a complete intercomparison of four coils are given. ("Proc.," Cambridge Phil. Soc., Vol. XVII., p. 340.)

ABSTRACT.

The following experiments were shown :—

1. Harmonic motion of a rigid body suspended by a torsion wire.
2. Determination of gravity by a rigid pendulum.
3. Determination of Poisson's ratio for indiarubber.
4. Two methods of measuring the surface tension of spherical soap films.
5. A simple viscometer for very viscous liquids.
6. Experiments with a prism of small angle.
7. Revolving table method of determining the curvature of spherical surfaces.
8. Experiments illustrating flare spots in photography.
9. Determination of the effective aperture of a photographic lens.
10. The comparison of nearly equal electrical resistances.

* It is not necessary, as a rule, to shunt both A and B, but it is simpler to suppose them both shunted in deriving the formula. If B is not shunted the value of b is infinite.

DISCUSSION.

Mr. DUDDELL complimented Dr. Searle on the simplicity and ingenuity of the arrangements. Some apparatus of the older style was so complicated that students took longer to understand the apparatus than to understand the principles it was intended to illustrate.

Principal SKINNER mentioned that he had once determined Poisson's ratio for cork in a simple manner with an old champagne cork. From the markings of the cork the length and diameter of the compressed portion could be found. The cork was then boiled to restore it to its original shape, and from its change in dimensions one could determine Poisson's ratio roughly.

Mr. F. E. SMITH associated himself with Dr. Searle's remark that small corrections constituted the fun of physics, in addition to being, very often, of fundamental importance. He hoped that when Dr. Searle had finished his other press work he would turn his attention to a book on practical physics.

Mr. S. D. CHALMERS thought the experiment shown with the small prism brought out very clearly the connection between a prism and a lens. In practice, when measuring the aperture of a photographic lens it was not convenient to use a large hole as done by Dr. Searle, as the mount cut off some of the sloping rays, and one had to do the best possible with a small hole.

Dr. RUSSELL said that personally he was much indebted to Dr. Searle, who had given him valuable assistance many times in the past.

Dr. SEARLE thanked the various speakers for their remarks.

IX. *The Vacuum Guard Ring and its Application to the Determination of the Thermal Conductivity of Mercury.*

By H. REDMAYNE NETTLETON, *B.Sc., Assistant Lecturer in Physics at Birkbeck College.*

1. *Introduction.*

DURING a research recently described before this society on the thermal conductivity of mercury by a method of impressed velocity, the author had occasion to employ a wide vacuum-jacketed tube. ("Proc." Phys. Soc., London, Vol. XXVI., December, 1913.) Though the tube was unsilvered and unprotected by any non-conductor of heat, the reduction of emissivity loss was so marked that calculation showed the possibility, in spite of the relatively small value of the conductivity of mercury, of obtaining, within a similar vessel in which the mercury was at rest, a temperature gradient sufficiently constant over a range of several centimetres to enable a direct determination of the thermal conductivity to be made.

The chief disadvantage of the ordinary guard ring lies in the uncertainty of the area from which the heat is collected, an uncertainty which becomes more pronounced if means other than the ice calorimeter be employed to measure the quantity of heat transmitted. The efficient substitution of a vacuum for a guard ring would remove the objection and open up the way for further development of both electrical and continuous-flow methods of calorimetry.

The measurement, by the steady flow of water, of the quantity of heat transmitted by conduction was first carried out by Callendar and Nicolson ("Proc." Inst.C.E., 1898) in their experiments on a cast-iron bar. Consistent results were obtained, and the calorimetry was very satisfactory, but the temperature gradient down the bar being far from linear its value near the entrance to the calorimeter, where some distortion of the isothermals is likely, had to be deduced by interpolation. Again, Searle ("Phil. Mag.," Jan., 1905, p. 125) has used the continuous flow method in his elegant laboratory apparatus for the determination of the thermal conductivity of copper. The bar being some 4 cm. in diameter and well protected by non-conducting material, the assumption is made that the temperature gradient is sufficiently linear to allow a measurement of it to be made from two thermometers 7·5 cm. apart. The thermal conductivity of mercury, however, is

only about one-fiftieth that of copper. None the less it is shown below that by virtue of the vacuum surrounded with cotton wool the heat lost in a length of some 12 cm. must be a negligible fraction of the quantity of heat transmitted. The measurement of temperature gradient can thus be very accurately effected with the aid of a single thermo-junction carried by a cathetometer, and at the same time the horizontal nature of the isothermals can be readily demonstrated, while the quantity of heat transmitted by conduction can be measured by a modification of Searle's method.

The determination of the thermal conductivity of mercury by a method so direct and simple when once the vacuum vessel has been constructed would seem warranted by the great difference in the values obtained by different experimenters for a substance for which the other constants are known to a particularly high accuracy.

It is hoped that the usefulness of the vacuum guard ring will not be limited to the case of mercury, for it is possible that it may afford useful information with respect to the relative conductivities of other liquids; more particularly experiment indicates that it is likely to prove useful for determining the thermal conductivity of metallic solids.

2. *Description of Apparatus and Method of Experimenting.*

(*a*) *The Vacuum Vessel.*—The specially constructed vacuum vessel is seen at V in Fig. 1. The inner tube is of uniform cross-section about 4·9 cm. in diameter and 40 cm. long. The outer tube, about 20 cm. long, is sealed at both its extremities to the inner one. Thus, there is no free end, as in an ordinary vacuum vessel, to permit of differences of contraction on cooling between two portions, but some elasticity is afforded by making the outer vessel concertina-shaped or corrugated. Owing to this double sealing, combined with the large width of tubing, some difficulty was experienced in obtaining such a vessel, which was at length supplied to the author from Germany early in 1913 by Messrs. Müller, Orme & Co. The vessel is silvered, and was exhausted to a high standard on a Gaede molecular pump.

The part of the inner tube not vacuum-jacketed is fitted with a long and annular heater, A. The lower part is fitted with a wide rubber cork, R, which carries the heat collecting spiral of copper S wound round the turned-down portion of the brass disc D. This disc, which was supported on the rubber

cork by a piece of ebonite let into both, was of a diameter slightly less than that of the inner tube of the vacuum vessel, and was filed in one place to permit the thermo-electric carrier tube T to pass it; it serves to collect the heat with a minimum

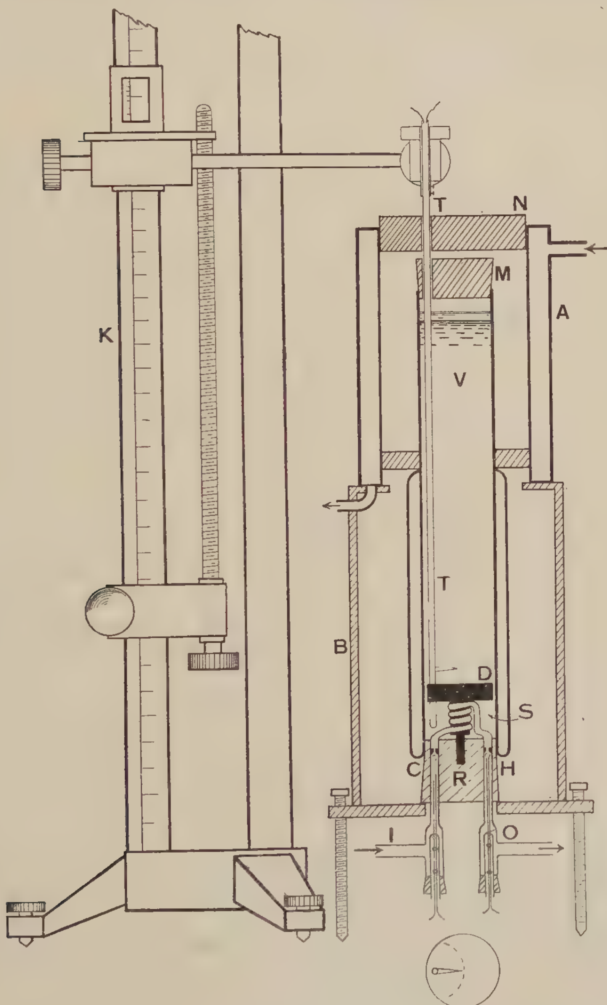

FIG. 1.

of convection, and at the same time provides a horizontal surface approximately isothermal. Both the brass disc and copper spiral were insulated with enamel to protect them from

the action of the mercury. The cork R was well secured by letting first seccotine and subsequently club enamel down on to the edges of it by means of a glass tube passing through the filed cavity in the brass disc. The vacuum vessel thus adapted for experiment was supported within the specially constructed box B fitted with levelling screws and made in two halves which could be locked together. The portion of the tube vacuum-jacketed was well wrapt round with cotton wool and filled the rest of the box.

The essence of an experiment is to fill the vacuum vessel with mercury, heat it at the top by circulating steam or other vapour through the annular heater A, determine the temperature gradient in the vacuum-jacketed portion *ev*, and measure the quantity of heat transmitted downwards by determining the difference of temperature between water entering and leaving the copper spiral. The mercury was covered with a layer of tar which effectively damped vibration, and greatly added to the steadiness of the temperature conditions obtained. Corks, M and N, closed the vacuum vessel and annular heater respectively, at the same time permitting the movement of the carrier tube T.

(*b*) *The Calorimetry.*—The calorimetry was effected with an iron-constantan thermo-couple giving 44·2 micro-volts per degree Centigrade, and situated at H and C. Early experiments showed differences of $3\frac{1}{2}$ per cent. in the value obtained for K, the thermal conductivity of mercury depending upon the magnitude of the water flow. As the limitation of the method was thus found to lie in the calorimetry great pains was taken to improve it. As now arranged the calorimetry, although still limiting the method, is consistent to about 1 per cent. The chief difficulty encountered in continuous flow calorimetry lies not so much in diminishing loss of heat by radiation as in ensuring the proper mixture of the outflowing water, for it is necessary that the higher steady temperature registered is the true average temperature over a cross-section of the warmed water. The vertical portion of the copper spiral is undoubtedly helpful in attaining this object, but further great improvement is effected in the following manner: The glass tube O holding the warm junction was plugged at its upper extremity with copper foil through four or five narrow channels in which the water was compelled to flow; close underneath the common outflow was the warm junction insulated and subsequently tipped at its head with copper foil, the head thus closely fitting

the glass tube. This arrangement proved very satisfactory, and probably the limit of accuracy in the calorimetry is determined by the conduction of heat across the rubber cork from the hot to the cold stream. The flow of water which was air-free by exhaustion was controlled by a constant head of sufficient height to force it through the fine channels in the copper foil, and through a greater resistance consisting of a drawn out capillary of glass, the height of which could be varied. The

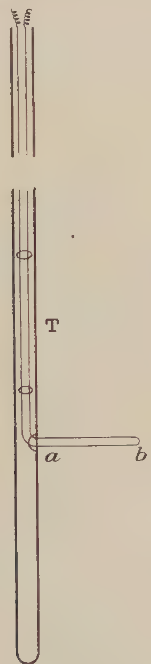

FIG. 2.

constancy of temperature of the inflowing water was much improved by passing it through a "condenser" consisting of a copper tube surrounded by a wider glass tube which extended well up the inlet tube I. (Fig. 1), and conveyed ordinary tap water, which in time became very steady.

(*c*) *The Measurement of Temperature Gradient and the Cold Junction Thermostat.*—The measurement of the temperature gradient within the portion of the mercury protected by the vacuum was effected by a single insulated iron-constantan thermo-junction. The thin insulated wires were contained

within the straightened vertical glass carrier tube T (Figs. 1 and 2), of less than 6 mm. diameter, and were fused through the glass at *a* (Fig. 2) with inappreciable external distortion, the wires from *a* to the junction *b* being close together, and in the same horizontal plane. The length *ab* was such that when the tube T was resting against the inner wall of the vacuum vessel the point *b* would just reach the centre of the latter. The thermo-electric carrier tube T was carried by the cathetometer K (Fig. 1), specially fitted with complete translational and rotational adjustments. The thermo-junction could thus be raised or lowered vertically through any measurable distance

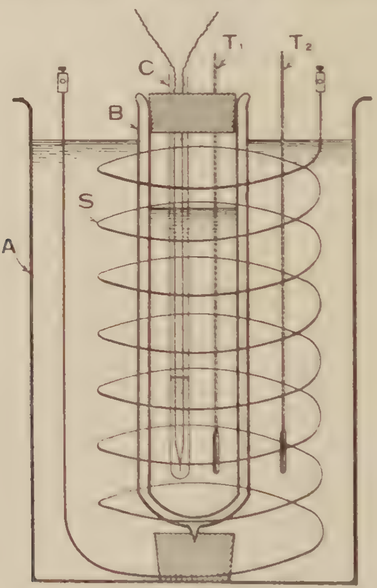

Fig. 3.

or rotated over a horizontal cross-section till the point *b* touched the walls of the vacuum vessel.

It will be seen that it is often necessary to maintain the other or "cold junction" of this circuit used for measuring the gradient at a steady temperature near the lower limit of the range measured by the hot one. The thermo-electric "thermostat" seen in Fig. 3 proved very efficient, the temperature certainly remaining constant to within 0·01°C. over the time of each independent experiment. A is a large outer vessel surrounded with cotton wool, and containing the

Dewar cylindrical vacuum vessel B, supported on a cork as shown. The thermo-junction is enclosed within the small glass tube C containing paraffin oil, and passing through the cork D. A and B are filled with water to the levels shown at a temperature approximate to that required, the water in B being covered with a layer of Fleuss pump oil. T_1 and T_2 are thermometers reading to 0·1°C. S is a coil of thick constantan wire lying between B and the walls of A. By passing a suitable current through S the temperature of the thermometer T_2 can be kept approximately constant at the desired value. When steady T_2 is observed at intervals of one hour or so, and the current, if necessary, is raised or lowered by means of a step rheostat. By this means, the mass of water being large, T_2 usually keeps constant to 0·1°C., showing that though the temperature may vary somewhat in different parts of A the temperature at any one place changes very slowly. The small temperature oscillations in A, however, are so damped by the vacuum that change of temperature in T_1 can often not be detected for hours. T_1 is uaually about half a degree below T_2, and very uniform throughout its mass. This arrangement is not only useful for holding a "cold junction," but may be used as a constant temperature source when calibrating thermo-couples in micro-volts per degree or when maintaining sodium sulphate at its transition point. With practice it can be very readily adjusted.

(*d*) *Method of Comparing the Two Fundamental Temperature Differences.*—Experiment shows that the temperature gradient within the portion of the mercury which is vacuum-jacketed may be regarded as constant. Let A be the effective cross-section of the vacuum vessel, and φ the rise of temperature in a space of 1 cm. Let θ be the difference of temperature on the same scale between the inflowing and outflowing water, the rate of flow being m grams per second. Then, taking the specific heat of water as unity we have for K the thermal conductivity of mercury:

$$KA\varphi = m\theta,$$

or,

$$K = m/A \times \theta/\varphi.$$

The ratio of two temperature intervals is thus required.

Now, as the relation between temperature and E.M.F. is very strictly linear for the iron constantan couple over the operative range 15°C. to 45°C., and as over this range the micro-volts per degree for both the calorimetric and gradient thermo-couples were found to agree to 1 part in 650, it is clear that it is only

necessary to compare the E.M.F. given by the two sets of couples. For this purpose both deflection and potentiometer methods were employed each having its special advantage.

While employing the deflection method the resistances of the two thermo-couples were adjusted to equality by suitably choosing the position of join of the two constantan wires of the calorimetric couple. This could be easily effected to an accuracy of 0·01 ohm and tested at any moment, and as an additional 30 to 50 ohms was always in circuit with the couples and galvanometer, change of resistance with rise of temperature on moving the gradient thermo-junction was quite negligible. The deflection produced by each circuit on inserting the proper

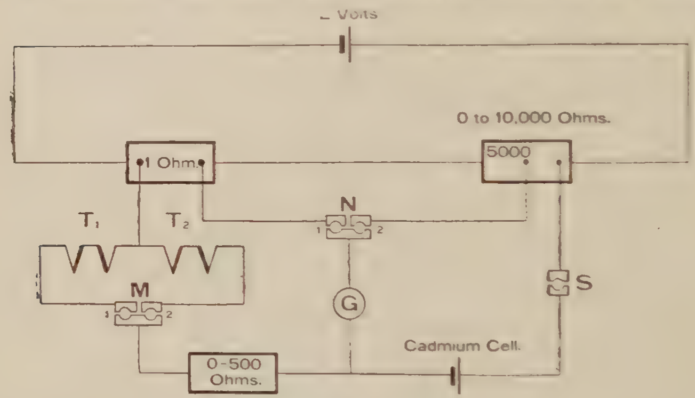

FIG. 4.

plug key was then proportional to the temperature interval. This was carefully verified directly by testing both couples simultaneously for one and the same temperature difference.

While the deflection method was very rapid and sufficiently accurate for experiments on the conductivity of mercury, yet, owing to the necessary limitation of the galvanometer scale, it was found inferior to the potentiometer method when attempting to test the exact nature of the temperature gradient or when measuring temperature differences directly in degrees Centigrade. In the absence of a potentiometer suitable for thermo-electric work, the arrangement shown in Fig. 4 was found very satisfactory. Briefly, the E.M.F. of either thermo-couple could be balanced against that across a standard ohm in series with an accumulator and a standard box of 0 to 10,000 ohms fitted with a dividing plug. The reciprocal of

the total resistance needed for a balance is, therefore, proportional to the temperature interval being measured as long as the total external voltage remains constant. This voltage can be measured in a moment by dividing the standard box into two portions, one of 5,000 ohms and the other slightly less, and such that the E.M.F. across it exactly balances that of a standard cell. The micro-volts given by either couple, and hence the interval in degrees Centigrade, can thus be easily calculated if required. A slight simplification of the connections reduces them to those necessary for the deflection method. Typical results illustrative of both methods are given in the next section. Precisely the same arrangement served for measuring in degrees Centigrade the difference of temperature between the inflowing and outflowing water in the experiments on the electrical supply of heat to the vacuum vessel, which are described in section 4.

Every care was taken to minimise local thermoelectric effects. All four joins of iron to copper wire were in small tubes side by side contained within a large test tube holding paraffin oil, and immersed in a Newton's annular cooling calorimeter filled with water, an arrangement which in previous work had proved satisfactory. Brass plug keys for joining copper wires were tested, and found suitable both from the thermoelectric and resistance standpoints. The keys should be large, and the plug holes well away from the copper joins. Both couples registered satisfactory zeros when the hot and cold junctions were placed side by side in the same stirred water, while the slight disturbance which was caused by deliberately touching the keys with the hand at the join of the two metals quickly subsided on removing the fingers. Repeatedly removing and inserting the actual plugs caused no appreciable effect. Joins of copper, brass and manganin can easily be rendered thermo-electrically safe, but with other metals special care has to be exercised.

3. *Results of Experiments.*

(*a*) *Horizontal Nature of Isothermals in Mercury.*—It will be seen from figures given in sub-section (*b*) below that a temperature gradient of just over 3°C. per centimetre was obtained over a length of some 12 cm. when the vacuum vessel filled with mercury was heated at the top by steam. On rotating the thermo-junction over a horizontal section within this range there was no appreciable alteration of galvanometer deflection even when the junction was on the point of touching the walls,

and the protruding wires were lying along the side of an inscribed hexagon. The only exception to this is to be found in the 2 cm. immediately above the disc D at the lower " end " of the column. This is illustrated by the figures in the next column, which show that the distortion is of an unsymmetrical character, and accounts for the slightly greater value of the temperature gradient seen in Tables III. and IV. of subsection (*b*) in the same region.

TABLE I.—*Showing Nature of Isothermals.*

Cathetometer reading.	Relative Temperatures with thermo-junction lying at the positions :—		
	Extreme left.	Middle.	Extreme right.
65	22·20	21·78	21·50
66	9·30	9·05	8·95
67	−3·60	−3·57	−3·55
68	−16·20	−16·20	−16·16

Thus, in the short space of 2 cm. the horizontal surfaces have become appreciably isothermal. A perhaps equally rigorous proof of this is afforded by the fact that repeated measurements of temperature gradient made over ranges of 7 cm. or 8 cm. in the middle of the vessel never differed by so much as 1 part in 500, whether the thermo-junction was moved vertically through the middle of the column or when almost touching the walls.

(*b*) *The Linear Nature of the Temperature Gradient in Mercury.* The first experiment performed with the apparatus at once revealed the linear nature of the temperature gradient. Boiling alcohol was used to supply the heating vapour, and no tar was covering the mercury surface. Deflections on the galvanometer scale are proportional to temperatures, the actual temperatures at the cathetometer positions marked 66 and 77 being approximately 23·4° C. and 46·2° C. respectively.

TABLE II.—*Showing Relative Temperatures at Definite Cathetometer Positions by the Deflection Method.*

Cathetometer reading.	Deflection.	Cathetometer reading.	Deflection.	Differences per 6 cm.
65	23·35	71	8·25	15·10
66	20·88	72	5·68	15·20
67	18·42	73	3·08	15·34
68	15·88	74	0·68	15·20
69	13·35	75	−1·93	15·28
70	10·75	76	−4·45	15·20
71	8·25	77	−7·00	15·25
72	5·68	78	−9·25	14·93

This result obtained by the deflection method was more than confirmed when the potentiometer method illustrated by the connections in Fig. 4 was employed. Table III. gives the figures for an actual experiment, the reciprocal of the total ohms needed for a balance being proportional to the difference of temperature between the hot and cold thermo-junctions. The surface of the mercury was covered with tar and steam was used as the heating vapour. The cold junction in the thermostat was at 25·43°C., the accumulator was balanced for the Weston cell at 4,848 ohms (viz., 4,846 ohms before the first reading and 4,850 ohms after the last one), and hence at any position of the hot junction where R is the total number of ohms needed for a balance the actual temperature in degrees Centigrade is given by :—

$$\theta = 25{\cdot}43 + \mathrm{W} \times \frac{10^6}{44{\cdot}2} \times \frac{9{,}849}{4{,}848} \times \frac{1}{\mathrm{R}},$$

where W is the E.M.F. of the Weston cell. Thus, the actual temperatures at cathetometer positions 67 and 76 are 36·61°C. and 64·69°C. respectively which give for the mean value of temperature gradient a value of 3·12°C. per centimetre. In the table below, however, it suffices to give actual readings and relative temperatures :—

TABLE III.—*Showing Relative Temperatures at Definite Cathetometer Positions by the Potentiometer Method.*

Cathetometer position.	Total ohms for balance.	Recripocal of total ohms × 10^7	Differences.	Differences per 6 cm.
66	5,824	1,717·0	...	...
67	4,185	2,389·5	672·5	...
68	3.270	3,058·1	668·6	...
69	2,684	3,725·8	667·7	...
70	2,277	4,391·7	665·9	...
71	1,977	5,058·2	666·5	...
72	1,748	5,720·8	662·6	3,331
73	1,565	6,389·8	669·0	3,332
74	1,417·0	7,057·2	667·4	3,331
75	1,295·0	7,722·0	664·8	3,330
76	1,192·2	8,387·9	665·9	3,329

The closeness of agreement between the figures in the last column is in part accidental, as the decimal places for the total ohms in the cathetometer positions 74 to 76 were obtained by proportional deflections, and for the other positions the balance

was only found to the nearest ohm. Apart from this each balance was very definite, and there is no doubt that the cathetometer can be set to an exact whole centimetre division with a greater accuracy than fractions can be read by the vernier. A repetition made two days later again gave remarkable closeness. The actual figures of this experiment are tabulated below :—

TABLE IV.—*Relative Temperatures at Cathetometer Positions.*

Cathetometer position.	Total ohms for balance.	Reciprocal of total ohms × 10^7	Differences.	Differences per 3 cm.
66	7,301	1,389·7	...	...
67	4,901	2,040·4	670·7	...
68	3,693	2,707·8	667·4	...
69	2,961	3,377·2	669·4	2,008
70	2,473	4,043·7	666·5	2,003
71	2,122·0	4,712·5	668·8	2,005
72	1,858·6	5,380·4	667·9	2,003
73	Omitted not	observed.	...	...
74	1,488·1	6,720·0	669·8	2,008

The linear character of the temperature gradient was also seen in the direct experiments on the thermal conductivity of mercury described in the next sub-section. It would seem that by moving the thermo-junction, say, from position 67 to position 76 the temperature gradient can be measured to an accuracy approaching 1 part in 1,000.

(*c*) *Determination of the Thermal Conductivity of Mercury.*—A typical experiment using the deflection method is given below, the order being as follows : At an exact minute a weighed flask was placed in position to receive the outflowing water. The deflection in the calorimetric circuit corresponding to the difference of temperature between the inflowing and outflowing water was then observed. The temperature gradient was next measured by raising the thermo-junction, readings being usually taken at every half-centimetre between the cathetometer positions 67 and 70·5. The calorimetric deflection was again observed, and the flask removed at the next available whole minute. The temperature of the room was then recorded as well as the approximate temperatures of the hot and cold water, as read by thermometers lying in the inlet and outlet tubes I and O of Fig. 1. All readings could be made in an interval of 15 minutes.

TABLE V.—*Typical Deflection Experiment on the Thermal Conductivity of Mercury.*

α Measurement of temperature gradient.

Cathetometer reading.	Deflection.	Cathetometer reading.	Deflection.	Difference.
67·0	15·02	69·0	− 3·75	18·77
67·5	10·26	69·5	− 8·34	18·60
68·0	5·54	70·0	−13·12	18·66
68·5	0·90	70·5	−17·87	18·77

Mean difference, 18·70.
ϕ=Difierence per centimetre, 9·35

β other measurements.

Initial calorimetric deflection	21·05
Final calorimetric deflection	21·12
Mean value of θ	21·09
Temperature of inflowing water	17·1°C.
Temperature of outflowing water	24·1°C.
Temperature of room	20·3°C.
Temperature of " cold junction "	35·68°C.
Area of cross-section of mercury	18·47 sq. cm.
Magnitude of flow of water	$0{\cdot}160_5$ gms. per sec.

Whence $$K=\frac{0{\cdot}1605}{18{\cdot}47}\times\frac{21{\cdot}09}{9{\cdot}35}$$
$$=0{\cdot}0196_0 \text{ C.G.S. units.}$$

A typical potentiometer experiment is illustrated by the figures of Table VI. :—

TABLE VI.—*Typical Potentiometer Experiment.*

Initial calorimetric balance	10,670 ohms.
Gradient balance at position 71	2,103 ohms.
Gradient balance at position 67	4,406 ohms.
Final calorimetric balance	10,700 ohms.
Weston cell balance	4,853 ohms.
Magnitude of water flow	0·2400 gms. per sec.
Gradient cold junction	25·70°C.
Temperature of room	20·4°C.
Temperature of outflowing water	22·2°C. (approx.).
Temperature of inflowing water	17·6°C. (approx.).

Whence ϕ the temperature gradient$=\frac{1}{4}\left(\frac{1}{2103}-\frac{1}{4406}\right)$, and θ the calorimetric difference$=\frac{1}{10685}$, giving $K=0{\cdot}0195_7$ C.G.S. units.

The mean value obtained for K the thermal conductivity of mercury as the result of 24 experiments under varying flows is $0{\cdot}0196_0$ C.G.S. units, the range of temperature being 35°C. to 45°C. No results have been rejected in this sequence. The highest value obtained was $0{\cdot}0198_2$ and the lowest $0{\cdot}0194_2$, but 19 of the 24 values do not differ by as much as 0·5 per cent.

from the mean. The greatest flow of water employed was 19·14 grams per minute, and the slowest flow 9·5 grams per minute. Tables VII. and VIII. below summarise the results of these experiments :—

TABLE VII.—*Values of* K *by Deflection Experiments.*

m=flow of water in grms. per second.	θ=deflection of calorimetric circuit.	ϕ=mean change of deflection per cm. of gradient circuit.	dt=approx. difference of temp. between inflowing and outflowing water in deg. C.	K
$0{\cdot}223_7$	16·16	10·08	5·5	$0{\cdot}0194_3$
$0{\cdot}275_3$	13·28	10·06	4·7	$0{\cdot}0196_8$
$0{\cdot}160_5$	21·09	9·35	7·0	$0{\cdot}0196_0$
$0{\cdot}222_1$	16·31	$9{\cdot}92_5$	5·6	$0{\cdot}0197_6$
$0{\cdot}275_7$	13·48	10·15	4·7	$0{\cdot}0198_2$
$0{\cdot}314_7$	15·14	$13{\cdot}03_5$	4·1	$0{\cdot}0197_9$
$0{\cdot}255_5$	18·28	12·84	4·9	$0{\cdot}0196_9$
$0{\cdot}212_9$	22·24	13·08	5·5	$0{\cdot}0196_0$
$0{\cdot}210_4$	22·38	$13{\cdot}02_5$	5·5	$0{\cdot}0195_7$
$0{\cdot}177_2$	26·19	12·88	6·4	$0{\cdot}0195_1$
$0{\cdot}260_0$	18·64	$13{\cdot}32_5$	4·7	$0{\cdot}0196_9$
$0{\cdot}237_1$	20·18	13·25	5·1	$0{\cdot}0195_5$
$0{\cdot}206_0$	22·99	13·10	5·8	$0{\cdot}0195_7$

Mean value of $K = 0{\cdot}0196_4$.

TABLE VIII.—*Values of* K *by Potentiometer Experiments.*

m=flow of water in grms. per second.	θ=mean recip. ohms of cal. circuit $\times 10^7$.	ϕ=change of recip. ohms per cm. of gradient circuit $\times 10^7$.	dt=approx. difference of temp. of hot and cold water in deg. C.	K
$0{\cdot}240_0$	935·6	621·2	4·6	$0{\cdot}0195_7$
$0{\cdot}232_2$	1,026	660·2	5·1	$0{\cdot}0195_4$
$0{\cdot}247_2$	986·9	677·4	4·8	$0{\cdot}0195_0$
$0{\cdot}319_1$	792·1	696·2	4·0	$0{\cdot}0196_6$
$0{\cdot}183_3$	1,294	658·6	6·2	$0{\cdot}0195_0$
$0{\cdot}217_2$	1,117	667·8	5·5	$0{\cdot}0196_8$
$0{\cdot}214_6$	1,126	669·4	5·5	$0{\cdot}0195_4$
$0{\cdot}255_0$	939·8	663·0	4·4	$0{\cdot}0195_7$
$0{\cdot}221_0$	1,070	658·8	5·0	$0{\cdot}0194_3$
$0{\cdot}168_8$	1,370	639·2	6·5	$0{\cdot}0195_8$
$0{\cdot}158_3$	1,451	636·2	6·7	$0{\cdot}0195_5$

Mean value of $K = 0{\cdot}0195_6$.

(*d*) *Experiments on Temperature Gradient and Isothermals in Water.*—Numerous experiments were performed on the temperature gradient and the isothermals in water, but it is perhaps out of place in this Paper to go into detail. Suffice it here to say that the gradient experiments confirm the efficacy of the vacuum guard ring. With respect to the isothermals in water,

they, too, are approximately horizontal over the cathetometer range 76 to 70, where very satisfactory measurements of temperature gradient can be made. But distortion of a non-symmetrical character is very pronounced in the neighbourhood of the collecting disc D (Fig. 1), and persists, though in gradually diminishing degree, till the thermo-junction has been raised four or five centimetres.

(e) *Determination of Area of Cross-section of the Vacuum Vessel.*—The mean area of cross-section of the operative part of the vacuum vessel could easily be found to an accuracy of 1 part in 1,000, with the aid of mercury and an electrical point contact carried by the cathetometer used previously for measuring the gradient. The vessel is remarkably uniform over the 12 cm. previously traversed by the thermo-junction, the mean area of cross-section over any particular length of 5 cm. taken not differing from that over any other similar length by as much as 1 part in 500. The mean area of cross-section at 20°C. is 18·74 sq. cm. The cross-section of the thermo-junction carrier tube was 0·27 sq. cm., and hence the effective mercury area is 18·47 sq. cm.

4. *Electrical Supply of Heat to the Vacuum Vessel.*

During the tedious process of improving the calorimetry it was very useful to be able to supply to the vacuum vessel an amount of heat generated electrically at a rate approximately equal to that transmitted down the vessel when full of mercury and heated at the top by steam. By this means the constancy of the heat collected under different rates of water flow could rapidly be tested. Further, the better the design of the heater, from the point of view of preventing heat loss from the top of the vacuum vessel, the more valuable does it become as a check on the reliability of the continuous flow calorimetry.

The heater (Fig. 5) consists of slightly less than 2 metres of thin manganin wire, gauge 34, wound as nearly as possible in a horizontal plane over fine saw cuts in an annular fibre frame, F, of about 4·2 cm. diameter. This frame is supported by a bridge and rod of fibre, the latter being let in to the cork cylinder C which was turned down till of a diameter such as to fit the vacuum vessel very closely. The ends of the manganin heating coil were soldered to thin copper wires of gauge 32, which on leaving the frame were brought up along slits in the cork for about 10 cm., where they were again joined to thicker copper wire of gauge 22, which led to the double terminals

T, T on the fibre handle. The cork cylinder C reached beyond the top of the vacuum vessel when the frame F was just touching the brass disc. The heating coil and leads, which were of about 16 ohms resistance, were insulated with two or three applications of club enamel, and dried at a high temperature by the electric current; they were then varnished with

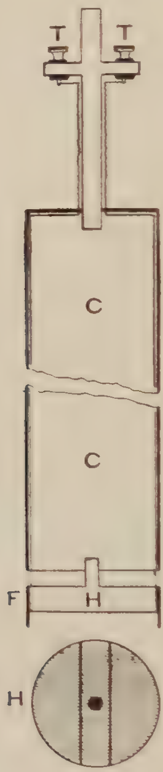

Fig. 5.

velure and again dried. Thus treated the heating coil could safely be immersed in mercury.

When using the heater a little mercury was poured into the vacuum vessel till it covered the horizontal collecting disc (Fig. 1) to a depth of about 5 mm. The heater was then pushed down the vacuum vessel till the bottom of the fibre-ring lay on this disc, and displaced sufficient mercury to completely cover the insulated coil. A constant current derived

from accumulators of 0·55 ampere as registered on a 0·0—0·6 Paul ammeter was then passed through the heater, while the heat was collected and measured by flowing water, the same arrangement as in Fig. 4, serving to measure the difference of temperature.

The heat generated by the electric current between the terminals of the heater was measured to an accuracy of 0·5 per cent., use being made of the copper voltameter, taking the proper precautions, to measure the value of the current, which would thus be correct to about 1 part in 500. The resistance of the heater and external leads were measured on a Post Office box on which they could be switched by the same key that broke the main current. The resistance of the heater between its terminals was found to be 15·93 ohms when cold and 15·95 ohms when tested isntantly after breaking the heating current.

It will be realised that the copper wires soldered on to the manganin coil were so chosen in gauge that while the heat generated in them electrically was small the heat conducted away by them was also inappreciable. It is easy to show the heat lost by conduction in an infinite wire of perimeter p and area of cross-section a in a uniform enclosure and dipping into a constant temperature source at U degrees above the enclosure is, when conveying electric current, less than a quantity $H=U\sqrt{EpKa}$ calories per second, where E is the coefficient of emissivity loss and K the thermal conductivity of the wire. Moreover, U, the temperature of the mercury layer touching the cork, can be roughly estimated from a knowledge of the distance between the heating coil and the collecting disc and the value of the temperature gradient, known from the conductivity experiments, which will permit the passage downwards of the heat generated and collected. In this way it is estimated that the heat lost per second by the two wire leads of gauge 32 is of the order 0·005 calorie per second, which is almost as great as that lost by the entire cork protected by the vacuum jacket. The resistance of the leads within the terminals, but outside the mercury was 0·07 ohm, so that the rate of generation of heat within them is again 0·005 calorie per second. It would seem likely, then, that there is a loss of quite 1 per cent. of the heat generated by the electric current between the terminals of the heater.

The following are some of the results obtained: A constant current of $0{\cdot}546_3$ ampere was passed through the heater of resistance 15·95 ohms between the terminals. The rate of

generation of heat, taking J at 20°C. as 4·180 joules, is thus 1·139 calories per second. The heat collected is seen from the following table :—

TABLE IX.—*Showing Heat Collected by the Calorimeter from an Electrical Heater.*

m=rate of flow of water in grms. per second.	$d\theta$=temp. difference between hot and cold water.	$m \times d\theta$=quantity of heat collected per second.
$0{\cdot}301_0$	$3{\cdot}70_9$	1·116
$0{\cdot}261_0$	$4{\cdot}23_2$	1·105
$0{\cdot}243_7$	$4{\cdot}55_2$	1·109
$0{\cdot}242_0$	$4{\cdot}60_7$	1·115
$0{\cdot}204_7$	$5{\cdot}41_2$	1·108
$0{\cdot}162_1$	$6{\cdot}83_2$	1·107

The mean of 12 consecutive experiments gives 1·110 calories per second as the rate at which heat is collected, which thus falls short of the heat generated electrically by just over 2·5 per cent. Estimating, then, 1 per cent. at least as the loss of heat from the top of the vessel—a loss of course, which does not enter into account in the main conductivity experiments—there would seem a possibility of an absolute error of 1·5 per cent. in the calorimetry of the method. But it must be realised that the electrical test, though interesting, and, perhaps, satisfactory, cannot be regarded as a rigorous electrical calibration of the flow calorimetry employed—for the small percentage of heat lost from the top is an uncertain quantity.

5. *Conclusion.*

The value of the thermal conductivity of mercury—viz., $0{\cdot}0196_0$ C.G.S. units over the range 35°C. to 45°C.—may be compared with the values obtained by other methods which are given in the table below :—

TABLE X.

Experimenter.	Method.	Temp.	Value of K.
H. F. Weber, 1880	Flat plate	17°C.	0·0162
Angström, 1864	Periodic flow of heat	50°C.	0·0177
Berget, 1888	Ordinary guard-ring	0°C.-100°C.	0·0202
R. Weber, 1903	Electrical measurement of heat	0°C.-34°C.	0·0197
The Author, 1913	Impressed velocity	15·5°C.	0·0201

The present result is thus confirmatory of the high values obtained by "steady state" methods, as against the much lower, but, perhaps, more usually accepted values obtained by the "variable state" methods of Angström and H. F. Weber.

The author's belief is that the thermal conductivity of mercury is not far below the value 0·020, and, if anything, slightly increases with the temperature between, say, 30°C. and 50°C. In view of the importance attached by such authorities as Lord Kelvin and Callendar to periodic flow methods, and in view of the fact that Angström's experiments were performed as far back as 1864, prior, that is to say, to the development of electrical thermometry of precision, it might seem desirable to repeat some of the older work, introducing modern improvements.

Incidentally, the vacuum guard-ring has proved a success, and the accuracy with which temperature gradients can be measured within it would seem to warrant the development of a system of calorimetry better suited for measuring heat supplied at so small a rate. The other possibilities of such a guard ring have already been mentioned.

The author would record his appreciation of the interest taken in his work by the Head of the Physics Department, Dr. A. Griffiths, who has been specially considerate in his endeavours to obtain for the Author so much of the apparatus needed.

ABSTRACT.

A specially constructed vacuum vessel heated at the top by steam and cooled at the bottom by flowing water, is used to find the thermal conductivity of mercury. The vacuum acts as a guard ring, which is at the same time not open to the well-known objection of communicating to the calorimeter a quantity of heat difficult to estimate. So efficient is the vacuum that the temperature gradient as measured by a single thermo-junction carried by a cathetometer is probably not in error to the extent of 1 part in 500.

The calorimetry is effected by the continuous-flow method on the lines suggested by Searle in his well-known laboratory apparatus for finding the thermal conductivity of copper. The rate of supply of heat, however, is only about 1 calorie per second, and the space in which it is desirable to collect it is necessarily somewhat limited. It is thus only after considerable trouble that the conditions have been obtained which yield a consistency of about 1 per cent. between experiments performed on different days or on the same day under varying rates of flow. The reliance to be placed on the calorimetry is greatly confirmed by a series of tests made with a specially-designed electric heater. The mean value obtained for the thermal conductivity of mercury in a set of 24 experiments is $0{\cdot}0196_0$ C.G.S. units over the range 35°C. to 45°C. The remarkable linear nature of the temperature gradient obtained within the vessel, the cross-section of which was very uniform, over the larger range of temperature 35°C. to 65°C. would indicate at least that there is no diminution of thermal conductivity with rise of temperature.

All temperature measurements were made with iron constantan

thermo-couples, special care being taken to eliminate local thermo-electric effects. A simple arrangement is described for keeping a "cold junction" during the time of an experiment constant to 0·01° C. at temperatures above the room.

DISCUSSION.

The AUTHOR has communicated the following reply to points raised by various Fellows with whom he discussed the apparatus after the meeting: In general thermo-couples made of wires taken off the same reel do not agree closely in thermo-E.M.F. But junctions made of pure charcoal, iron wire and constantan taken off the same reels, the junctions being welded, and all other joints being twisted with pliers and immediately painted, agree with remarkable closeness, if the usual precautions be taken to quell local thermo-electric effects. I have tested at least six such couples, including the two pairs used in the research, which agreed at the worst to 1 part in 650. At Dr. Griffiths' suggestion I calibrated prior to the meeting two more such couples between fixed points, obtaining to the nearest microvolt 1.427 and 1.428 microvolts between 0° C. and 32·383° C., the transition point of sodium sulphate and 4.424 and 4.426 microvolts respectively between 0°C. and 99·64°C. Dr Griffiths and others witnessed the definiteness of the balances on the Cambridge thermo electric potentiometer. I have just discovered that Palmer ("Phys. Review," XXI., 1905) records similar agreement. Unfortunately, constantan is a very uncertain material, being known under various names, such as advance, climax, eureka, the small change in percentage composition causing large effects in the thermo-electric properties: a previous specimen I tested gave 50 microvolts per degree Centigrade. I regret I can form no estimate of the small quantity of heat lost per second from the sides of the vacuum vessel. While the experiments on water greatly confirm the efficacy of the guard ring no concrete conclusion can be drawn from these, for it is impossible to say how much of the slight fall in temperature gradient is due to emissivity, and how much due to the rise of thermal conductivity of water with fall of temperature.

X. *Practical Harmonic Analysis.* *By* ALEXANDER RUSSELL, *M.A., D.Sc.*

RECEIVED NOVEMBER 30, 1914.

Introductory.

IF the value of a function, $f(x)$, recur over successive intervals, λ, of the variable x, so that the equation $f(x+n\lambda)=f(x)$ is true for all integral values of n, then Fourier showed that we may write

$$f(x)=a_0+a_1\cos(2\pi/\lambda)x+a_2\cos 2(2\pi/\lambda)x+\ \ldots$$
$$+b_1\sin(2\pi/\lambda)x+b_2\sin 2(2\pi/\lambda)x+\ \ldots \quad (1)$$

where a_0, a_1, b_1, . . . are constants which he expressed in the form of definite integrals. [*See* (2) given below.] He pointed out* that these coefficients were as real constants of a periodic curve as, for instance, its area or its centroid. Even when we neglect the higher harmonics in (1) it is advisable to use the Fourier values of the coefficients, as it is known† that, assuming the ordinary law of probability, these values make the sum of the mean squares of the errors a minimum.

In the "Philosophical Magazine" for August, 1874, p. 95, J. O'Kinealy showed how the theorem (1) quoted above follows at once from the ordinary symbolical methods used in solving differential equations.

Taylor's theorem for the expansion of a function of two variables may be written as follows :—

$$f(x+\lambda)=\varepsilon^{\lambda\frac{\partial}{\partial x}}f(x),$$

where ε is the base of Napierian logarithms. Hence, we may write the equation $f(x+\lambda)=f(x)$ in the form

$$(\varepsilon^{\lambda\frac{\partial}{\partial x}}-1)f(x)=0.$$

Regarding this as a differential equation, the roots of the auxiliary equation are given by

$$\varepsilon^{\lambda p}=1,$$

where p has been written for $\partial/\partial x$. Now, by De Moivre's theorem,

$$\varepsilon^{\lambda(2m\pi/\lambda)\iota}=\cos 2m\pi+\iota\sin 2m\pi=1,$$

where ι stands for $\sqrt{-1}$. Hence $p=m(2\pi/\lambda)\iota$, and m can have

* Joseph Fourier, "Theory of Heat," Freeman's translation, p. 198.
† C. H. Lees, "Proc." Phys. Soc., London, Vol. XXVI., p. 275, 1914

any positive or negative integral value. Therefore, by the rules given for solving differential equations, the general solution is

$$f(x)=a_0+a_1'\varepsilon^{(2\pi/\lambda)x\iota}+a_2'\varepsilon^{2(2\pi/\lambda)x\iota}+\ldots$$
$$+b_1'\varepsilon^{-(2\pi/\lambda)x\iota}+b_2'\varepsilon^{-2(2\pi/\lambda)x\iota}+\ldots$$
$$=a_0+a_1\cos(2\pi/\lambda)x+a_2\cos 2(2\pi/\lambda)x+\ldots$$
$$+b_1\sin(2\pi/\lambda)x+b_1\sin 2(2\pi/\lambda)x+\ldots$$

And if $f(x)$ be real, the constants are all real. It will be seen that this is Fourier's theorem. Fourier also gave the following exact mathematical values for these constants :—

$$a_0=\frac{1}{\lambda}\int_0^\lambda f(x)\partial x\,;\quad a_n=\frac{2}{\lambda}\int_0^\lambda f(x)\cos n\left(\frac{2\pi x}{\lambda}\right)\partial x\,;$$

and $$b_n=\frac{2}{\lambda}\int_0^\lambda f(x)\sin n\left(\frac{2\pi x}{\lambda}\right)\partial x. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

The integral for b_n was previously given by Lagrange.* When we know the mathematical expression for $f(x)$, approximate numerical values of the constants can generally be determined without much difficulty.

In certain cases when the function is discontinuous difficulties arise in interpreting the meaning of the results obtained at the points where discontinuities occur, but this, although a point of great interest to the mathematician, does not affect the practical usefulness of the theorem and so we do not discuss it here.

In engineering practice we are, as a rule, given the graph of $f(x)$, and we have to determine the constants a_0, a_1, b_1, . . . The value of a_0 can be found at once by finding the area $\int_0^\lambda f(x)\partial x$ and dividing by λ. The area must be determined by some method of mechanical quadrature, several of which are in everyday use. It is to be noticed that this gives us an approximate value of the integral. In one method a certain number of ordinates are measured and the integral is expressed in terms of them. In exactly the same way we can find the values of a_n and b_n in terms of certain selected ordinates of the curves

$$y=f(x)\cos n(2\pi/\lambda)x \text{ and } y=f(x)\sin n(2\pi/\lambda)x$$

respectively, it being unnecessary to draw the graphs of these

* *See* Todhunter's "Integral Calculus," Chap. XIII.

curves, as the values of the ordinates can be found at once by multiplying $f(x)$ by the required cosine or sine coefficient. Before showing the best way of doing this, we shall give a brief *résumé* of the methods ordinarily employed.

Practical Methods.

There are three typical methods used in everyday work. In the first method all harmonics the order of which is higher than some value n are neglected. In this case the equation $y=f(x)$ contains $2n+1$ constants, and so we require to know the values of at least $2n+1$ ordinates in order to determine the constants. We get $2n+1$ equations of the form

$$\begin{aligned} y_1 = a_0 + a_1 \cos(2\pi/\lambda)x_1 + a_2 \cos 2(2\pi/\lambda)x_1 \ldots \\ + b_1 \sin(2\pi/\lambda)x_1 + b_2 \sin 2(2\pi/\lambda)x_1 \ldots \end{aligned}$$

And many methods* have been given for lightening the labour involved in solving these equations for a_0, $a_1, b_1, \ldots$ and so finding the equation to a curve $y=f(x)$ having the same $2n+1$ ordinates as the given curve.

C. F. Gauss ("Werke," Vol. III., p. 281) showed how we could write down the required equation at once in the following form

$$\begin{aligned} y = {} & y_1 \frac{\sin\frac{1}{2}(x-x_2)\sin\frac{1}{2}(x-x_2)\ldots\sin\frac{1}{2}(x-x_{2n+1})}{\sin\frac{1}{2}(x_1-x_2)\sin\frac{1}{2}(x_1-x_3)\ldots\sin\frac{1}{2}(x_1-x_{2n+1})} \\ & + y_2 \frac{\sin\frac{1}{2}(x-x_1)\sin\frac{1}{2}(x-x_3)\ldots\sin\frac{1}{2}(x-x_{2n+1})}{\sin\frac{1}{2}(x_2-x_1)\sin\frac{1}{2}(x_2-x_3)\ldots\sin\frac{1}{2}(x_2-x_{2n+1})} \\ & + \ldots \qquad (3) \end{aligned}$$

where x_1, x_2, ... are the abscissæ corresponding to the ordinates y_1, y_2, ...

To prove (3) it is sufficient to notice that when $x=x_1$, $y=y_1$, when $x=x_2$, $y=y_2$, &c., and that the coefficients of y_1, y_2, ... when expanded contain only sines and cosines of integral multiples of x which are not greater than n.

It is, however, laborious to find the coefficients of $\cos mx$ and $\sin mx$ from (3), and thus the theorem is not of much use for harmonic analysis, although it is of value in interpolation.

Apart altogether from the labour involved in the above group of methods, a serious drawback is the lack of any

* F. W. Grover, Bureau of Standards "Bull.," 9, p. 567, 1914. For Runge's method, *see* Gibson's "Introduction to the Calculus," Chapter XI.

indication as to how far the calculated values of the harmonics differ from their true Fourier values. If n is large it is probable that the values of a_m and b_m found in this way are approximately correct when m is small; but if m be equal or nearly equal to n it is highly probable that the values are quite different from the Fourier values.

In the second method an attempt is made to determine the values of the definite integrals given in (2) by mechanical quadrature. For this purpose we have found that Weddle's rule, which we discuss below, is the most suitable. One of its advantages is that the value of the amplitude and phase of any given harmonic can be determined separately from the other harmonics to any required degree of accuracy. The lower the order of the harmonic the less is the arithmetical labour involved in finding its value.

In the third method certain infinite series are obtained connecting the values of the Fourier constants, the orders of which are odd multiples of n. When the harmonics diminish rapidly in value this method is a simple and easy one.*

We show below how to obtain many other series of a like nature, the use of which extends the range and increases the accuracy of the method.

The final conclusion we arrive at is that the method of evaluating the Fourier integrals by mechanical quadrature is by far the best. The series formulæ, however, are of value, especially as supplementary aids in checking the computed results.

Weddle's Rule.

The rule given by Thomas Weddle† for finding the area A included between a curve, two ordinates and the axis of x ($BPQN_6O$ in Fig. 1) is

$$A = \int_0^x y\delta x = \frac{x}{20}[y_0 + y_2 + y_4 + y_6 + 5(y_1 + y_5) + 6y_3] \quad . \quad (4)$$

where $ON_1 = N_1N_2 = \ldots = N_5N_6 = x/6$, and $y_0, y_1, \ldots$ are the values of the ordinates at the points $O, N_1, \ldots$

* Fischer-Hinnen, "Elektrotechnik und Maschinenbau," Vol. XXVII., p. 335, 1909. S. P. Thompson, "Proc." Phys. Soc., Vol. XXIII., p. 334, 1911.

† "On a New and Simple Rule for Approximating to the Area of a Figure by means of seven Equidistant Ordinates," "The Cambridge and Dublin Mathematical Journal," p. 79, February, 1854.

The formula was obtained by the method of finite differences. The following proof, however, is simpler, more accurate and more instructive.

First of all the assumption is made that the equation to the curve can be put in the form

$$y=a_0+a_1x+a_2x^2+\ .\ .\ .\ +a_nx^n \quad . \quad . \quad . \quad . \quad . \quad (5)$$

This curve can be made to pass through $n+1$ points on the curve BPQ (Fig. 1), and it is the simplest curve that can be

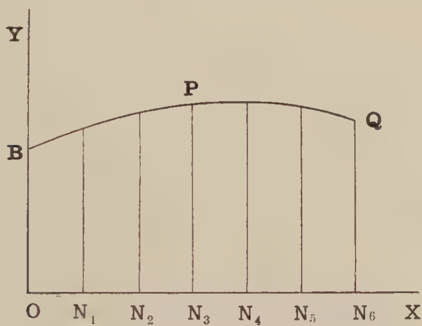

FIG. 1.—WEDDLE'S RULE.

drawn through these $n+1$ points. Its equation may be written in the form

$$\begin{aligned} y=&\,y_1\frac{(x-x_2)(x-x_3)\ .\ .\ .\ (x-x_{n+1})}{(x_1-x_2)(x_1-x_3)\ .\ .\ .\ (x_1-x_{n+1})} \\ &+y_2\frac{(x-x_1)(x-x_3)\ .\ .\ .\ (x-x_{n+1})}{(x_2-x_1)(x_2-x_3)\ .\ .\ .\ (x_2-x_{n+1})} \quad . \quad . \quad . \quad . \quad (6) \\ &+\ .\ .\ . \end{aligned}$$

By expanding the coefficients of y_1, y_2, . . . in this equation and comparing with (5) the values of a_0, a_1, a_2, . . . are at once found.

By applying Weddle's rule to (5) we find that the term a_mx^m adds to the value found for the area the expression

$$\frac{a_mx^{m+1}}{20.6^m}[2^m+4^m+6^m+5(1+5^m)+6.3^m] \quad . \quad . \quad . \quad . \quad (7)$$

and the term a_0 adds a_0x to the value of the area. Hence,

L 2

putting $m=1, 2, \ldots$ in (7) and simplifying we find that Weddle's rule gives the following value for the area :—

$$\begin{aligned} A = a_0x + a_1(x^2/2) + a_2(x^3/3) + a_3(x^4/4) + a_4(x^5/5) \\ + a_5(x^6/6) + a_6(x^7/7)(1{\cdot}00013) \\ + a_7(x^8/8)(1{\cdot}00051) + a_8(x^9/9)(1{\cdot}00145) \\ + a_9(x^{10}/10)(1{\cdot}00343) + a_{10}(x^{11}/11)(1{\cdot}00691) + \ldots \quad (8) \end{aligned}$$

By the integral calculus the true value of the area is

$$A = \sum_{p=0}^{p=n} a_p x^{p+1}/(p+1).$$

We see that, provided that n is not greater than 5, Weddle's formula is absolutely correct, and even for values of n as great as 10 the error *for individual terms* is well under 1 per cent.

For example, let us determine the value of $\int_0^{0\cdot6} \sinh x\partial x$ by (4). We get

$$\begin{aligned} \int_0^{0\cdot6} \sinh x\partial x = \frac{3}{100} [\sinh (0{\cdot}2) + \sinh (0{\cdot}4) + \sinh (0{\cdot}6) \\ + 5\{\sinh (0{\cdot}1) + \sinh (0{\cdot}5)\} + 6 \sinh (0{\cdot}3)] \\ = 0{\cdot}185\ 465\ 2 \ldots \end{aligned}$$

which is correct to the last figure. On the other hand, we see by (8) that the error in the value of $\int_0^x x^{10}\partial x$ found by quadrature is 0·69 of 1 per cent. It is advisable, therefore, to have means of checking the computed values.

Series Formulæ Involving Areas.

It is easy to show by trigonometry that

$$\begin{aligned} \cos x - \tfrac{1}{3} \cos 3x + \tfrac{1}{5} \cos 5x - \ldots \\ = \pi/4 \text{ from } x=(2n-1/2)\pi \text{ to } (2n+1/2)\pi \\ = -\pi/4 \text{ from } x=(2n+1/2)\pi \text{ to } (2n+3/2)\pi, \end{aligned}$$

and that

$$\begin{aligned} \sin x + \tfrac{1}{3} \sin 3x + \tfrac{1}{5} \sin 5x + \ldots = \pi/4, \text{ from } x=2n\pi \text{ to } (2n+1)\pi \\ = -\pi/4, \text{ from } x=(2n+1)\pi \text{ to } (2n+2)\pi. \end{aligned}$$

Hence, by Fourier's formulæ (2), we find that

$$a_1-\frac{a_3}{3}+\frac{a_5}{5}-\ .\ .\ .\ =\frac{2}{\lambda}\int_0^\lambda f(x)\left\{\cos\frac{2\pi x}{\lambda}-\tfrac{1}{3}\cos 3\frac{2\pi x}{\lambda}+\ldots\right\}\partial x$$

$$=\frac{\pi}{2\lambda}\left[\int_0^{\lambda/4}-\int_{\lambda/4}^{3\lambda/4}+\int_{3\lambda/4}^{\lambda}\right]y\partial x.$$

The series, therefore, on the left-hand side equals $(\pi/2\lambda)$ times the difference between the sum of the areas of the curve from 0 to $\lambda/4$ and from $3\lambda/4$ to λ, and the area of the curve from $\lambda/4$ to $3\lambda/4$.

Similarly, we see that

$$a_m-\frac{a_{3m}}{3}+\frac{a_{5m}}{5}-\ .\ .\ .\ =\frac{\pi}{2\lambda}\left[\int_0^{\lambda/4m}-\int_{\lambda/4m}^{3\lambda/4m}+\int_{3\lambda/4m}^{5\lambda/4m}-\ldots\right]y\partial x\ .\ (9)$$

and $$b_m+\frac{b_{3m}}{3}+\frac{b_{5m}}{5}+\ .\ .\ .\ =\frac{\pi}{2\lambda}\left[\int_0^{\lambda/2m}-\int_{\lambda/2m}^{2\lambda/2m}+\int_{2\lambda/2m}^{3\lambda/2m}-\ldots\right]y\partial x.\ (10)$$

It will be noticed that the expressions on the right-hand side of (9) and (10) represent areas which can be found by any of the ordinary methods. Formulæ (9) and (10), therefore, enable us to find approximate values for the Fourier coefficients when the amplitudes of the higher harmonics are small.

Let us first consider the case of the rectangular wave shown

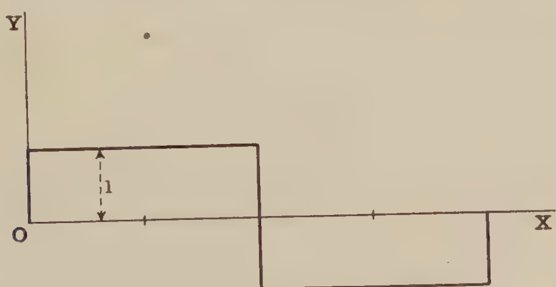

FIG. 2.—RECTANGULAR WAVE.

in Fig. 2. In this case y is $+1$, when x lies between 0 and $\lambda/2$, and -1 when x lies between $\lambda/2$ and λ.

By (9) we find that

$$a_m-a_{3m}/3+a_{5m}/5-\ldots=0,$$

for all values of m, and thus the coefficients of all the cosine terms are zero. By (10) we find that

$$b_m+b_{3m}/3+b_{5m}/5+\ldots=\pi/(2m),\ \text{or}\ 0,$$

according as m is odd or even.

If we neglect all harmonics whose orders are higher than 16, we have

$$b_1+b_3/3+b_5/5+\ldots=\pi/2$$
$$b_3+b_9/3+b_{15}/5+\ldots=\pi/6$$
$$b_5+b_{15}/3+\ldots=\pi/10,$$
$$b_7+\ldots=\pi/14, \text{ \&c.}$$

And hence we easily compute the values of b_1, b_3. . .

Harmonics.	Computed values.	True values.	Harmonics.	Computed values.	True values.
b_1	1·286	1·273	b_9	0·174	0·141
b_3	0·445	0·424	b_{11}	0·143	0·116
b_5	0·279	0·255	b_{13}	0·121	0·098
b_7	0·224	0·182	b_{15}	0·105	0·085

The true values of b_1, b_3. . . have been computed from the coefficients of the sine terms in the following well-known equation to the wave shown in Fig. 2—

$$y=\frac{4}{\pi}\left[\sin x+\frac{1}{3}\sin 3x+\frac{1}{5}\sin 5x+\ldots\right],$$

the wave-length being 2π.

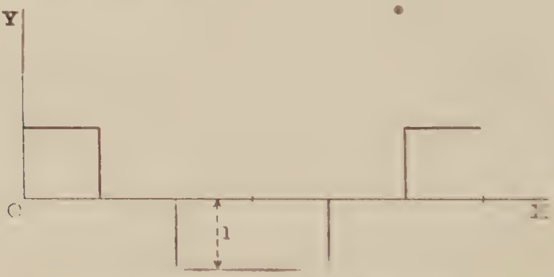

FIG. 3.—DISCONTINUOUS RECTANGULAR WAVE.

As the true values of the areas have been substituted in the formulæ used, the errors are due to the neglect of the seventeenth and higher harmonics. It will be seen that the error in the value of b_1 is about 1 per cent., and the errors in the computed values of b_7 and higher harmonics are greater than 23 per cent.

As a further example, let us consider the hightly irregular wave shown in Fig. 3.

The equation to this wave is

$$y=\frac{2\sqrt{3}}{\pi}\left[\cos x-\frac{1}{5}\cos 5x+\frac{1}{7}\cos 7x-\frac{1}{11}\cos 11x+\ldots\right].$$

In this case

$$a_1-\frac{a_3}{3}+\frac{a_5}{5}-\ldots=\tfrac{1}{4}\left[\int_0^{\pi/2}-\int_{\pi/2}^{3\pi/2}+\int_{3\pi/2}^{2\pi}\right]y\partial x,$$

$$=\int_0^{\pi/2} y\partial x=\frac{\pi}{3},$$

$$a_3-\frac{a_9}{3}+\frac{a_{15}}{5}-\ldots=0$$

$$a_5-\frac{a_{15}}{3}+\frac{a_{25}}{5}-\ldots=-\frac{\pi}{15}$$

$$a_7-\frac{a_{21}}{3}+\frac{a_{35}}{5}-\ldots=\frac{\pi}{21}, \text{ \&c.}$$

We also have $b_m+b_{3m}/3+b_{5m}/5+\ldots=0$ for all values of m. From the equations given above we see that $b_m=0$. Neglecting a_m when m is greater than 24, we easily compute the numbers given in the following table :—

Harmonics.	Computed values.	True value.	Harmonics.	Computed values.	True values.
a_1	1·100	1·103	a_{13}	0·081	0·085
$-a_5$	0·209	0·221	$-a_{17}$	0·062	0·065
a_7	0·150	0·158	a_{19}	0·055	0·058
$-a_{11}$	0·095	0·100	$-a_{23}$	0·046	0·048

The error in the value of a_1 is about 0·3 of 1 per cent., and in the value of the higher harmonics, which do not vanish, about 5 per cent.

Series Formulæ involving Ordinates (Thompson's Method.)

We have from (2)—

$$a_1+a_3+a_5+\ldots=\frac{2}{\lambda}\int_0^\lambda f(x)\left\{\cos\frac{2\pi x}{\lambda}+\cos 3\frac{2\pi x}{\lambda}+\ldots\right\}\partial x,$$

and hence, integrating by parts, we get

$$a_1+a_3+a_5+\ldots=\frac{1}{\pi}\left[f(x)\left\{\sin\frac{2\pi x}{\lambda}+\tfrac{1}{3}\sin 3\frac{2\pi x}{\lambda}+\ldots\right\}\right]_0^\lambda$$

$$-\frac{1}{\pi}\int_0^\lambda f'(x)\left[\sin\frac{2\pi x}{\lambda}+\tfrac{1}{3}\sin 3\frac{2\pi x}{\lambda}+\ldots\right]\partial x.$$

Noticing that the first expression on the right hand side vanishes at both limits, and that the series inside the bracket under the integral sign equals $\pi/4$ from 0 to $\lambda/2$ and $-\pi/4$ from $\lambda/2$ to λ, we get

$$\begin{aligned} a_1+a_3+a_5+\ldots &= -(1/4)[f(\lambda/2)-f(0)-f(\lambda)-f(\lambda/2)], \\ &=(1/2)[f(0)-f(\lambda/2)], \\ &=(1/2)(y_0-y_{\lambda/2}). \end{aligned}$$

In general, we have

$$a_m+a_{3m}+a_{5m}+\ldots=(1/2m)[y_0-y_{\lambda/2m}+y_{2\lambda/2m}-\ldots-y_{(2m-1)\lambda/2m}] \quad (11)$$

and

$$b_m-b_{3m}+b_{5m}-\ldots=(1/2m)[y_{\lambda/4m}-y_{3\lambda/4m}+y_{5\lambda/4m}-\ldots-y_{(4m-1)\lambda/4m}] \quad (12)$$

To illustrate the use of formulæ (11) and (12) let us consider the rectangular wave shown in Fig. 2. By (11) we see that all the coefficients of the cosine terms are zero, and by (12) if we neglect, as formerly, the 17th and higher harmonics we get

$b_{15}=1/15$, $b_{13}=1/13$, $b_{11}=1/11$, $b_9=1/9$, $b_7=1/7$, $b_5-b_{15}=1/5$, $b_3-b_9+b_{15}=1/3$, and $b_1-b_3+b_5-\ldots-b_{15}=1$. Hence, we find the computed values in the following table :—

Harmonics.	Computed values.	True values.	Harmonics.	Computed values.	True values.
b_1	1·224	1·273	b_9	0·111	0·141
b_3	0·378	0·424	b_{11}	0·091	0·116
b_5	0·267	0·255	b_{13}	0·077	0·098
b_7	0·143	0·182	b_{15}	0·067	0·085

We see that the error in the computed value of the first harmonic due to the method is greater than 4 per cent., and in b_7 and higher harmonics it is greater than 21 per cent. The accuracy obtained, therefore, in computing the values of b_7, b_9, .. by this method is in this case slightly greater than that obtained by the method of areas. In the case of b_7, for instance, this proves that the neglect of $-b_{21}+b_{35}-\ldots$ produces a smaller error than the neglect of $b_{21}/3+b_{35}/5+\ldots$ when using the method of areas. If consecutive terms are of opposite signs it will be seen that the method of areas would determine the amplitudes of all the harmonics much more accurately.

Let us now consider the wave shown in Fig. 3. In this case

the difficulty arises as to the value we are to assign to the ordinate when the abscissa has the values $\pi/3$, $2\pi/3$, $4\pi/3$. ... We see that if the value of x is a little less than $\pi/3$, y is 1, but if it is greater than $\pi/3$, y is 0. It would seem reasonable, therefore, to take $(0+1)/2$ as the value of y and a rigorous mathematical proof of this can be given.

From (12) we can show that $b_m=0$, and from (11) we get that $a_{3m}=0$, and

$$a_1+a_5+a_7+a_{11}+\ldots=1.$$

$$a_5+a_{25}+\ldots=-1/5,\ a_7=1/7,\ a_{11}=-1/11,\ a_{13}=1/13,\ldots$$

Hence, neglecting the 25th and higher harmonics, we get the computed values in the following table :—

Harmonics.	Computed values.	True values.	Harmonics.	Computed values.	True values.
a_1	1·121	1·103	a_{13}	0·077	0·085
$-a_5$	0·200	0·221	$-a_{17}$	0·059	0·065
a_7	0·143	0·158	a_{19}	0·053	0·058
$-a_{11}$	0·091	0·100	$-a_{23}$	0·044	0·048

The error in the value of a_1 is about 2 per cent., and in the value of a_5 and the higher harmonics it is greater than 10 per cent.

Comparison of the Series Formulæ Methods.

The waves analysed above are discontinuous, and the analysis shows that there is an infinite series of harmonics, the amplitudes of which are smaller the higher their order. The amplitudes of the harmonics, however, diminish slowly as their order increases. In both the examples considered above the amplitude of the 101th harmonic, for instance, is about 1 per cent. of the amplitudes of the fundamental. They, therefore, put the series methods to a severe test. We shall now consider waves which, although they are discontinuous and have an infinite number of harmonics, yet approach roughly in shape to a sine wave.

Let us consider, for instance, the wave which has the trapezoidal shape shown in Fig. 4. If the length of the straight line forming the top of the wave be $\lambda/6$, it is easy to show by Fourier's method that the equation to the curve shown in Fig. 4 is

$$y=\frac{6\sqrt{3}}{\pi^2}\left[\sin\frac{2\pi x}{\lambda}-\frac{1}{5^2}\sin 5\frac{2\pi x}{\lambda}+\frac{1}{7^2}\sin 7\frac{2\pi x}{\lambda}-\ldots\right].$$

As the wave approximates in shape much more closely to a sine wave than the cases previously considered the amplitudes of the harmonics are smaller, and we shall, therefore, consider the effect that neglecting the 7th and higher harmonics has on the accuracy of our results.

Fig. 4.—Trapezoidal Wave.

Using the area method, we get by formulæ (9) and (10)

$$b_1+b_3/3+b_5/5=\pi/3\,;\ \ b_3=0, \text{ and } b_5=-\pi/(3\cdot5^2).$$

Hence, $b_1=1\cdot056$ $(1\cdot053)$, $b_3=0$, $b_5=-0\cdot0419$ $(-0\cdot0421)$, the numbers in the brackets being the true values.

If we use the ordinates method, we get by (11) and (12)

$$b_1-b_3+b_5=1\,;\ \ b_3=0 \text{ and } b_5=-1/5^2.$$

Hence, $b_1=1\cdot040$ $(1\cdot053)$ and $b_5=-0\cdot040$ $(-0\cdot0421)$.

Both methods show that all coefficients the orders of which

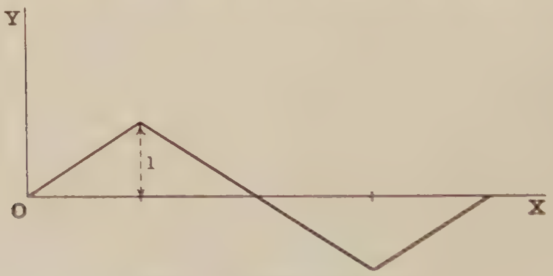

Fig. 5.—Triangular Wave.

are multiples of 3 vanish. By taking higher harmonics into account the accuracy of the ordinates method could be made quite satisfactory in this case, but a very large number of ordinates would have to be measured with high accuracy.

As a further example, let us consider the triangular wave.

shown in Fig. 5. By Fourier's method we find that the equation to this wave is

$$y=\frac{8}{\pi^2}\left\{\sin\frac{2\pi x}{\lambda}-\frac{1}{3^2}\sin 3\frac{2\pi x}{\lambda}+\frac{1}{5^2}\sin 5\frac{2\pi x}{\lambda}-\ldots\right\}.$$

The values of the ordinates and the areas required in formulæ (10) and (12) can easily be written down. The results of computing the harmonics by the two methods given above, when the 25th and higher harmonics are neglected are given in the following table :—

Harmonics.	True values.	Formula (10) (areas).	Formula (12) (ordinates).
b_1	0·8106	0·8106	0·8129
$-b_3$	0·0901	0·0901	0·0921
b_5	0·0324	0·0326	0·0356
$-b_7$	0·0165	0·0166	0·0181
b_9	0·0100	0·0097	0·0123
$-b_{11}$	0·0067	0·0065	0·0083
b_{13}	0·0048	0·0046	0·0059
$-b_{15}$	0·0036	0·0035	0·0044
b_{17}	0·0028	0·0027	0·0035
$-b_{19}$	0·0022	0·0022	0·0028
b_{21}	0·0018	0·0018	0·0023
$-b_{23}$	0·0015	0·0015	0·0019

In obtaining these results we assume that no less than 49 areas and 49 ordinates have been accurately measured. It will be seen that for all practical purposes the area method would be sufficiently accurate.

Accurate Formulæ.

As a general rule it is best to employ methods which aim directly at finding the values of the Fourier integrals. When the highest accuracy is desired it is essential to use these methods. One advantage they possess is that the amplitude and phase angle of each harmonic is determined independently of the others. In this case we can use the formulæ (9) to (12) to check the accuracy of our calculations.

To determine a_1 and b_1 accurately it is necessary to divide the wave-length λ into 24 equal parts, and measure the lengths of the ordinates $y_0, y_1, \ldots y_{23}$ at the ends and at the points of division, the ordinate y_{24} being equal to y_0, as the curve is periodic. By (2) we have

$$a_1=\frac{2}{\lambda}\int_0^\lambda y\cos\frac{2\pi x}{\lambda}\partial x,$$

$$=\frac{2}{\lambda}\left[\int_0^{\lambda/4}+\int_{\lambda/4}^{\lambda/2}+\int_{\lambda/2}^{3\lambda/4}+\int_{3\lambda/4}^{\lambda}\right]y\partial x.$$

Hence, applying Weddle's rule to each integral separately, we get

$$a_1=\frac{1}{40}\left[y_0\cos 0+y_2\cos 2\frac{2\pi}{24}+y_4\cos 4\frac{2\pi}{24}+y_6\cos 6\frac{2\pi}{24}\right.$$

$$+5\left(y_1\cos\frac{2\pi}{24}+y_5\cos 5\frac{2\pi}{24}\right)+6y_3\cos 3\frac{2\pi}{24}$$

$$\left.+y_6\cos 6\frac{2\pi}{24}+y_8\cos 8\frac{2\pi}{24}+\ldots\right]$$

$$=\frac{1}{80}\left[4y_0+y_4+y_{20}-(4y_{12}+y_8+y_{16})\right]$$

$$+\frac{\sqrt{3}}{80}\left[y_2+y_{22}-(y_{10}+y_{14})\right]$$

$$+\frac{1}{8}\left[\cos 15^0(y_1+y_{23}-y_{11}-y_{13})+\sin 15^0(y_5+y_{19}-y_7-y_{17})\right]$$

$$+\frac{3\sqrt{2}}{40}\left[y_3+y_{21}-y_9-y_{15}\right]. \quad \ldots \quad (13)$$

Similarly, we find that

$$b_1=\frac{1}{80}\left[y_2+4y_6+y_{10}-(y_{14}+4y_{18}+y_{22})\right].$$

$$+\frac{\sqrt{3}}{80}\left[y_4+y_8-(y_{16}+y_{20})\right].$$

$$+\frac{1}{8}\left[\sin 15^0(y_1+y_{11}-y_{13}-y_{23})+\cos 15^0(y_5+y_7-y_{17}-y_{19})\right.$$

$$+\frac{3\sqrt{2}}{40}\left[y_3+y_9-y_{15}-y_{21}\right]. \quad \ldots \quad (14)$$

Applying these formulae to the rectangular wave shown in Fig. 2, we get

$$a_1=0,$$

$$b_1=\frac{12}{80}+\frac{\sqrt{3}}{20}+\tfrac{1}{2}(\sin 15^0+\cos 15^0)+\frac{3\sqrt{2}}{10}$$

$$=1{\cdot}27324,$$

which is correct to the last figure.

The accuracy of the formula is thus of a high order. In everyday work we write $\sin 15^0=0{\cdot}259$ and $\cos 15^0=0{\cdot}966$ in (13) and (14).

Similarly, taking the same 24 ordinates, we get

$$a_2=\frac{1}{80}[4(y_0+y_{12}-y_6-y_{18})+y_2+y_{10}+y_{14}+y_{22}-(y_4+y_8+y_{16}+y_{20})],$$
$$+\frac{\sqrt{3}}{16}[y_1+y_{11}+y_{13}+y_{23}-(y_5+y_7+y_{17}+y_{19})] \quad . \quad (15)$$

$$b_2=\frac{\sqrt{3}}{80}[y_2+y_4+y_{14}+y_{16}-(y_8+y_{10}+y_{20}+y_{22})]$$
$$+\frac{1}{16}[y_1+y_5+y_{13}+y_{17}-(y_7+y_{11}+y_{19}+y_{23})]$$
$$+\frac{3}{20}[y_3+y_{15}-(y_9+y_{21})]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (16)$$

$$a_3=\frac{1}{40}[2y_0+y_8+y_{16}-(y_4+2y_{12}+y_{20})]$$
$$+\frac{\sqrt{2}}{16}[y_1+y_7+y_{17}+y_{23}-(y_5+y_{11}+y_{13}+y_{19})]$$
$$+\frac{3\sqrt{2}}{40}[y_9+y_{15}-(y_3+y_{21})]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17)$$

and

$$b_3=\frac{1}{40}[y_2+y_{10}+2y_{18}-(2y_6+y_{14}+y_{22})]$$
$$+\frac{\sqrt{2}}{16}[y_1+y_{11}+y_{17}+y_{19}-(y_5+y_7+y_{13}+y_{23})]$$
$$+\frac{3\sqrt{2}}{40}[y_3+y_9-(y_{15}+y_{21})]. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

To obtain the same accuracy for a_n as that given by formula (13) for a_1 we should have to measure 24 n ordinates. We should expect the accuracy of (17), for instance, not to be as high as that of (13), as we have taken the same number of ordinates in the two cases. Applying (17) and (18) to the rectangular wave (Fig. 2), we find that

$$a_3=0,$$
$$b_3=\frac{1}{40}(0)+\frac{\sqrt{2}}{16}(0)+\frac{3\sqrt{2}}{10}=0{\cdot}4243,$$

the true value being 0·4244.

The formulæ for the higher harmonics can be written down without difficulty, but naturally they are lengthy.

As a further example let us take the case of the circular wave, shown in Fig. 6. The wave-length is taken equal to 2.

If we only take 12 ordinates over the whole wave-length, then by Weddle's rule, owing to the symmetry of the wave,

$$10b_1 = 5y_{\lambda/12} + \sqrt{3}y_{2\lambda/12} + 6y_{3\lambda/12},$$

and thus $b_1 = 0{\cdot}5680$. If we take 24 ordinates we get $b_1 = 0{\cdot}56703$, and, finally, if we take 36 ordinates we get $b_1 = 0{\cdot}56703$. This, therefore, is the true value of b_1. In this

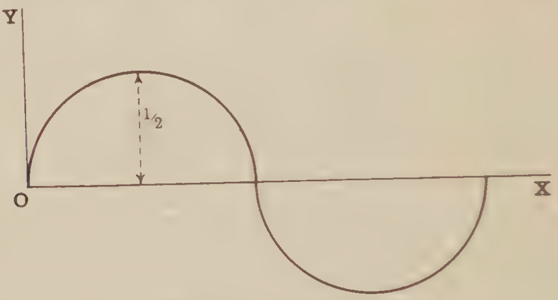

FIG 6.—CIRCULAR WAVE.

case it is easier to get the true value of the Fourier integral by the approximate mechanical rule than by analysis.

Special Formulæ.

In certain cases series formulæ for determining the harmonics become very simple.

In alternating-current work, for instance, the positive half of the wave is generally exactly similar to the negative half, and so $a_0 = 0$, and all the even harmonics vanish. If, in addition, $f(x) = f(\lambda/2 - x)$, formula (1) becomes

$$y = f(x) = b_1 \sin (2\pi/\lambda)x + b_3 \sin 3(2\pi/\lambda)x + \ldots$$

Hence we easily prove that

$$b_1 - b_{11} + b_{13} - b_{23} + \ldots + b_{12n-11} - b_{12n-1} + \ldots$$
$$= (y_{\lambda/12} + y_{\lambda/4})/3 + y_{\lambda/6}/\sqrt{3} \quad \ldots \quad (19)$$

and

$$b_5 - b_7 + \ldots + b_{12n-7} - b_{12n-5} + \ldots$$
$$= (y_{\lambda/12} + y_{\lambda/4})/3 + y_{\lambda/6}/\sqrt{3} \quad \ldots \quad (20)$$

Similarly, we can show that

$$b_1+\frac{b_{11}}{11}+\frac{b_{13}}{13}+\ \dots\ +\frac{b_{12n-11}}{12n-11}+\frac{b_{12n-1}}{12n-1}+\ \dots$$

$$=\frac{2\pi}{3\lambda}\left[2\int_0^{\lambda/4}-\int_0^{\lambda/6}\right]y\partial x+\frac{2\pi}{\lambda\sqrt{3}}\left[\int_0^{\lambda/4}-\int_0^{\lambda/12}\right]y\partial x\ .\quad .\quad (21)$$

and

$$\frac{b_5}{5}+\frac{b_7}{7}+\frac{b_{17}}{17}+\frac{b_{19}}{19}+\ \dots$$

$$=\frac{2\pi}{3\lambda}\left[2\int_0^{\lambda/4}-\int_0^{\lambda/6}\right]y\partial x-\frac{2\pi}{\lambda\sqrt{3}}\left[\int_0^{\lambda/4}-\int_0^{\lambda/12}\right]y\partial x\ .\quad (22)$$

We also have by (10) and (12)

$$b_3-b_9+b_{15}-\ \dots\ =\tfrac{1}{3}\left[2y_{\lambda/12}-y_{\lambda/4}\right]\ .\quad .\quad .\quad (23)$$

and

$$b_3+\frac{b_9}{3}+\frac{b_{15}}{5}+\ \dots\ =\frac{2\pi}{\lambda}\left[2\int_0^{\lambda/6}-\int_0^{\lambda/4}\right]y\partial x\ .\ (24)$$

These equations sometimes enable us to find b_1, b_3, . . . with very little trouble. For example, in the case of the trapezoidal wave (Fig. 4), we see at once that $\lambda=2\pi$, $\int_0^{\lambda/4}y\partial x=\pi/3$, $\int_0^{\lambda/6}y\partial x=\pi/6$ and $\int_0^{\lambda/12}y\partial x=\pi/24$.

Hence, by (21),

$$b_1+b_{11}/11+\ \dots\ =1{\cdot}053.$$

And thus neglecting small fractions of the amplitudes of higher harmonics of the orders $12n-1$ and $12n-11$ we get $b_1=1{\cdot}053$, which is correct to the last figure.

The equation to the circular wave (Fig. 6) is $y=\sqrt{x-x^2}$, and thus $y_{\lambda/12}=0{\cdot}3727$, $y_{\lambda/6}=0{\cdot}4714$ and $y_{\lambda/4}=0{\cdot}5000$. We also have

$$\int_0^{\lambda/12}y\partial x=0{\cdot}0430,\quad \int_0^{\lambda/6}y\partial x=0{\cdot}1146,\quad\text{and}\quad \int_0^{\lambda/4}y\partial x=0{\cdot}1964.$$

Hence, by measuring three ordinates and three areas only, we get by formulæ (19) to (24)

$$\begin{aligned}
&b_1-b_{11}+b_{13}-\ \dots && =0{\cdot}563\\
&b_5-b_7+b_{17}-\ \dots && =0{\cdot}019\\
&b_1+b_{11}/11+b_{13}/13+\ \dots && =0{\cdot}569\\
&b_5/5+b_7/7+b_{17}/17+\ \dots && =0{\cdot}013\\
&b_1-b_3+b_5-\ \dots && =0{\cdot}500\\
&b_3-b_9+b_{15}-\ \dots && =0{\cdot}088
\end{aligned}$$

From these equations we might deduce as a first approximation that $b_1=0{\cdot}569$ $(0{\cdot}567)$, $b_3=0{\cdot}088$ $(0{\cdot}094)$, &c., where the true values are given in brackets. With the possible exception of b_1, we have no guarantee of the accuracy of our results, as the equations show that the higher harmonics are not negligible, and since for a circular wave $f'(x)=\infty$, when x is zero there must be an infinite number of terms in the Fourier series. The true values found by Weddle's rule are $b_1=0{\cdot}567$, $b_3=0{\cdot}0942$, $b_5=0{\cdot}0400$, $b_7=0{\cdot}0252$, . . . and as only ordinates need to be measured, the accuracy of the data is higher than when areas have to be measured.

As a final example, let us consider the curve $y=x-2x^3+x^4$ (Fig. 7), which is almost indistinguishable from the sine curve,

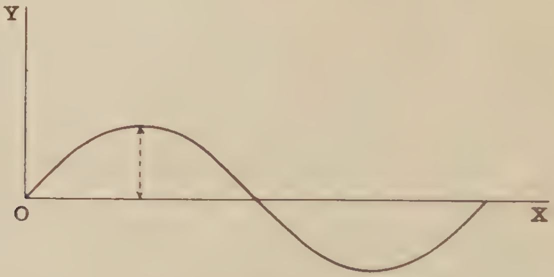

FIG. 7.—BIQUADRATIC WAVE.

whose equation is $y=(5/16)\sin \pi x$ for values of x between 0 and 1.

We have

$$y_{\lambda/4}=5/16,\ y_{\lambda/6}=22/81, \text{ and } y_{\lambda/12}=\frac{205}{1296}.$$

Hence, if we neglect the 11th and higher harmonics, we get by (19)

$$b_1=(y_{\lambda/12}+y_{\lambda/4})/3-y_{\lambda/6}/\sqrt{3}=0{\cdot}313\ 704,$$

the true value being $0{\cdot}313\ 705$. We also have

$$\int_0^{\lambda/4} y\,\partial x=\pi/10,\ \int_0^{\lambda/6} y\,\partial x=61\pi/1215 \text{ and } \int_0^{\lambda/12} y\,\partial x=263\pi/19440$$

and thus, by (21),

$$b_1=0{\cdot}313\ 705.$$

By Fourier's method we can show that the equation to the curve is

$$y=\frac{96}{\pi^5}\left[\sin\frac{2\pi x}{\lambda}+\frac{1}{3^5}\sin 3\frac{2\pi x}{\lambda}+\ .\ .\ .\right],$$

and so the values of the higher harmonics are extremely small. This explains the high accuracy attainable in this case by the series formulæ.

If the alternating-current wave be such that $f(x) =-f(\lambda/2-x)$, its equation must be of the form

$$y=f(x)=a_1\cos(2\pi/\lambda)x+a_3\cos 3(2\pi/\lambda)x+\ .\ .\ .$$

In this case we have

$$a_1-\frac{a_{11}}{11}+\frac{a_{13}}{13}-\frac{a_{23}}{23}+\ .\ .\ .=\frac{2\pi}{3\lambda}\left[\int_0^{\lambda/12}+\int_0^{\lambda/4}\right]y\partial x + \frac{2\pi}{\lambda\sqrt{3}}\left[\int_0^{\lambda/6}y\partial x\right]\quad .\ .\ (25)$$

and $$\frac{a_5}{5}-\frac{a_7}{7}+\frac{a_{17}}{17}-\frac{a_{19}}{19}+\ .\ .\ .=\frac{2\pi}{3\lambda}\left[\int_0^{\lambda/12}+\int_0^{\lambda/4}\right]y\partial x - \frac{2\pi}{\lambda\sqrt{3}}\left[\int_0^{\lambda/6}y\partial x\right].\quad .\ .\ (26)$$

For instance, if we take the wave-form shown in Fig. 3 and neglect $a_{11}/11$, &c., we get

$$a_1=\frac{\pi}{6}+\frac{\pi}{9}\sqrt{3}=1\cdot13\ (1\cdot10),$$

and $$\frac{a_5}{5}-\frac{a_7}{7}+\ .\ .\ .=-0\cdot081\ (-0\cdot067),$$

where the true values are given in the brackets. The large errors in this case are due to the very distorted shape of the wave analysed. For waves approaching cosine shape the accuracy attainable would be far higher.

Conclusion.

The results given above prove that it is best to determine the values of the Fourier constants $a_0, a_1, a_2, \ldots b_1, b_2, \ldots$ of a periodic curve directly by mechanical quadrature. The

particular method of quadrature which the author has found best is that first suggested by Weddle. In a few cases, especially when the function has a point or points of discontinuity, the accuracy of the formulæ given is not of the highest order, although sufficient for practical work. In these cases, if higher accuracy is desired, it is necessary to subdivide the Fourier integrals, taking the limits at the points of discontinuity and finding the values of the component integrals separately by quadrature.

The various series formulæ given in the Paper sometimes enable us to find the values of the lower harmonics very readily. Their main use, however, lies in checking the values of the Fourier constants found by approximate formulæ.

ABSTRACT.

Making the assumption that the graph of a periodic function is given, the problem of the best way of determining the Fourier constants in the series equation which represents it is considered. The ordinary method of procedure is to neglect all the harmonics above a certain order and determine the coefficients of the harmonic terms by making the curve represented by this equation pass through a number of arbitrarily selected points on the given curve. This is the method used, for instance, by Runge and Grover. A serious defect in this solution is that the values found for the amplitudes of the harmonics, more especially for the higher harmonics, may be very different from their true Fourier values. The method gives no indication of the magnitude of these errors. Gauss pointed out many years ago that the solution of this limited problem could be written down at once mathematically, and that it was of importance in certain interpolation problems in astronomy. Another method has been suggested recently by Silvanus Thompson. He uses certain series formulæ for finding the Fourier constants. The author suggests other series formulæ of a similar kind. If the given curve be approximately sine shaped so that the amplitudes of the higher harmonics are small, this method is both simple and accurate. For distorted waves, however, the lack of accuracy is serious in practice. It has also the drawback that an error made in computing the value of one of the constants may introduce errors in the computed values of others. The author gives numerical examples to illustrate the accuracy attainable by the use of infinite series formulæ. He concludes by pointing out that in the great majority of cases much the best method of procedure when determining the constants is to evaluate Fourier's integrals by the methods of mechanical quadrature given in books on the calculus of finite differences. In particular he has found that Weddle's rule is admirably adapted for the practical computation of the Fourier integrals. By means of this rule, a new and simple proof of which is given, each constant is determined separately to a high order of accuracy. Numerical examples are given to illustrate this. The series formulæ used by Thompson can

be usefully employed either for verifying the values found by mechanical quadrature or for indicating when the higher harmonics cannot be neglected.

DISCUSSION.

The President thought the best way of impressing on students the importance of Fourier's theorem was to begin with the fundamental and add the successive harmonics one at a time, thus showing how the resultant curve approximates more and more nearly to the analysed original. In the problem treated by the Author the periods were selected arbitrarily. Very often it was necessary in practical problems to determine both the coefficients and the periods of harmonic constituents. The inverse process was performed in mechanical integrating machines, such as Kelvin's tide predicting machine, in which the curve resulting from the integration of a number of harmonic constituents of known amplitude and period was obtained.

Prof. S. P. Thompson thought any method which would give increased accuracy in the determination of the higher harmonics was welcome. He had suggested in a recent Paper that one should begin with the higher harmonics and work backwards towards the lower. This had the defect that any error in the initial determinations tended to accumulate. He wished to challenge the accuracy of Weddle's rule. He applied it to the case of an isoceles triangle, the length of which was three times the base. Dividing it into six parts and applying Weddle's rule, the area is given as 1·6 (the base being 1) instead of 1·5. He thought this rather discountenanced Weddle's rule, except for very smooth curves. It had long been desirable to have some harmonic analyser which would pick out a particular period without previous knowledge as the ear does in the case of sound. This was done to a certain extent in Schuster's periodogram.

Mr. F. J. W. Whipple, criticising Prof. S. P. Thompson's example of the triangle, said that a triangle could not be regarded as the simplest curve through the six points chosen. If we have a smooth curve which is periodic, and apply Weddle's rule to the area included by a complete period divided by, say, six ordinates, we would get six separate values of the area if we start respectively at y_0, y_1, y_2 . . . Taking the mean of these we get the simple well-known expression

$$\tfrac{1}{2}(y_0+y_1+y_2+y_3+y_4+y).$$

He showed two slides showing a simple way by which an interpretation of a Fourier series, such as

$$\sin\theta+\sin 3\theta/3+\sin 5\theta/5\ .\ .\ .\ +\ \ .\ \&c.$$

is easily obtained.

The Author, in reply, agreed with the President that one of the best ways of impressing the meaning of Fourier's theorem on the mind of the student was to construct graphically the curves formed by adding in succession the various harmonics to the fundamental, thus illustrating how the resulting curves approximated more and more to the shape of the original wave analysed. Michelson and Stratton have constructed a machine for drawing these curves automatically. It is described in the "Phil. Mag." for 1898, and many curves illustrating its use are shown. They show, for instance, the curves obtained by adding together three terms, five terms, seven terms, twenty-one terms and seventy-nine terms respectively of the Fourier series for a rectangular wave. The last figure they obtain is almost indistinguishable from a rectangle. He also referred to Arthur Wright's device for finding the harmonics electrically. Prof. Thompson points out that when Weddle's rule is applied to an isosceles triangle the error is nearly 7 per cent. The

reason of the large error is that the vertex of a triangular wave is a point of discontinuity. As mentioned in the Paper, therefore, Weddle's rule ought to be applied over each half of the base separately. When this is done the correct answer is obtained. Disappointment at the lack of accuracy sometimes obtained when this rule is applied indiscriminately is doubtless responsible for its neglect by many mathematicians, notwithstanding the high commendation passed on it by Prof. Boole nearly 60 years ago. It has a sounder theoretical basis than any similar rule, and in conjunction with the series formulæ given in the Paper it affords a method in many cases, possibly the only method, of computing the Fourier constants with high accuracy. The Author much appreciated Mr. Whipple's graphical method of showing how the sum of Fourier's sine series for $\pi/4$ gradually approximated to this value.

XI. *Measuring the Focal Length of a Photographic Lens.* By T. Smith, *B.A.*

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The principal focus of a lens of focal length f is at a distance fF/f' from that of the combination of focal length F formed by placing in front of the first lens another of focal length f'. This suggests a simple method of finding the focal length of a photographic lens, which can be divided into two parts, each capable of producing a real image of a distant object. Let f and f' be the focal lengths of the two components, and F that of the complete lens. Set up the whole lens in the camera, and focus a distant object sharply on the ground glass. Now unscrew the front component of the lens from its mount without disturbing the rest of the lens, and measure the distance d through which the ground glass has to be moved for the same object to be sharply focussed by the back component used alone. Then

$$d=\frac{f\mathrm{F}}{f'}. \qquad (1)$$

Next, take the whole lens out of the camera, and insert it the other way round, so that what is usually the back component is now in front Focus as before with the complete lens for a distant object, and measure the displacement of the ground glass necessary to focus the same object when the component now in front is removed Denote this distance by d'.

Then

$$d'=\frac{f'\mathrm{F}}{f}, \qquad (2)$$

or, combining (1) and (2),

$$\mathrm{F}^2=d\,d'. \qquad (3)$$

This method avoids the difficulty of measuring exactly a transverse magnification, and also is not subject to errors arising from want of parallelism of object and image, from distortion and other oblique aberrations. When, as is often the case, the two components are equal, a single measurement suffices, the formula then reducing to

$$\mathrm{F}=d,$$

When the two components are unequal, the ratio of their focal lengths is determined by

$$f/f' = \sqrt{d/d'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The results (3) and (4) may be readily proved by noting that the positions in which the images are formed by the separate components are conjugate foci for the complete lens—viz., that pair of conjugate foci for which the beam of light between the two components is parallel.

When the separation of the two components of a lens combination can be increased by a known amount t, the focal length of either component can be found directly. Suppose that with this increased separation the focal length of the combination is F′, and the distance between its principal focus and that of the back combination alone is d''.

Then

$$\frac{1}{F}-\frac{1}{F'}=\frac{t}{ff'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

and

$$d''=\frac{fF'}{f'}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The elimination of F and F′ between (1), (5) and (6) gives

$$f^2\left(\frac{1}{d}-\frac{1}{d''}\right)=t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

for finding the focal length of the back component. This result again is at once obvious from the ordinary expression for longitudinal magnification, since the principal foci of the front component and the focussing screen are in all cases in the positions of conjugate foci for the back component.

For the measurement of d'', when the lens mount has no means of adjustment, the front component can be conveniently fixed at one end of a short tube of metal or cardboard, the other end of which slips over or into the ordinary mount.

When f has been found, equation (4) gives f', the focal length of the front component.

A number of interesting variations suitable for exercising the ingenuity of a student will suggest themselves, *e.g.*, to find the focal lengths of a compound lens, such as the "Tessar," and of its components, one of which is diverging, when another converging lens of unknown focal length is provided as an auxiliary.

ABSTRACT.

The focal length of a compound lens is obtained solely by focussing on the camera screen the image of a distant object on the lens axis

by the complete lens and by each of its components separately. One additional focussing of the same object when the separation of the components is altered determines the focal lengths of each component. The method is both accurate and quick, and requires only a camera and the lens.

DISCUSSION.

Dr. A. Russell thought the method was extremely neat and likely to become very widely used by those who had such measurements to perform.

Prof. G. H. Bryan, in a communication which was read by the Secretary, described a "rough and ready" method which he had used to find the focal length of a lens when nothing was available but a photograph taken with the lens of some suitable object, such as a rectangular building (Fig. B). The two infinity points of perspective i' and j' are found by producing the horizontal lines of the building until they meet. The centre X of the picture is found, and through it a line is drawn perpendicular to $i'j'$, cutting it at a point, h'. On $i'j'$, as hypotenuse, a right-angled triangle, $i'N'j'$, is constructed, with its apex, N′, on the perpendicular through X. Then, if X is on or near $i'j'$, $h'N'$ is the focal length of the lens. If the angle of the building is not 90°, but some other angle, θ, then angle $i'N'j'$ should also be θ. To get greater accuracy, with centre X and radius $h'N'$ cut $i'j'$ in Q. Qh' is then equal to the focal length.

The Author, in a partly communicated reply, commented on Prof. Bryan's construction, which, he said, was very neat so far as it went; but had the misfortune to be incorrect in general. It would not be difficult to find cases in which the result of measurements by this method would differ from the focal length of the lens employed by as much as 75 per cent. When the effect of the position of the stop, which limits the beam of light transmitted by the lens, upon the perspective of the projection of a solid object on a plane is adequately considered it will be seen that the length obtained by Prof. Bryan's construction is not necessarily the focal length of the lens, but is merely the distance from which the picture should be viewed to secure a correct impression of the view.

It may be noted in passing that the length obtained by this construction is liable to a number of errors which are frequently by no means negligible. Apart from those due to distortion, &c., the result would be affected by trimming the print unequally on the two sides, and considerable trimming is the rule rather than the exception. The more exact method mentioned appears to be incorrect even when Prof. Bryan's assumptions are adopted, for in taking such a photograph the plate would be vertical, the lens axis horizontal and the proper amount of subject included on the plate by employing the rising front of the camera. The perspective of the photograph of a solid object is determined, not by the focal length of the lens, but by the position of the stop which limits the light beams that are transmitted by the lens. In Fig. A let CDE represent in plan part of the exterior of the building of which a photograph is to be taken; let N be the position of the front nodal point of the lens, and L the centre of the effective limiting stop. If the actual stop is behind some of the component lenses L will be the (virtual) image of its centre in that part of the lens system which is situated in front of the stop. Assume that the lens gives an image of a plane object free from aberrations, and in particular from distortion, curvature and astigmatism. Let IHGJ be the plane focussed on and *ihgj* its image. The rays from a point, E, on the cottage, not in the plane focussed for, which eventually pass through the lens, fill a cone whose vertex is E and base a circle, centre L, parallel to the plane of the plate. This cone will meet the plane IHGJ in a circle, centre G, and what passes as the image of E on the plate is an image of this circle. If the stop is sufficiently small this circular image will be indistinguishable from a point, and we may for the purposes of this argument

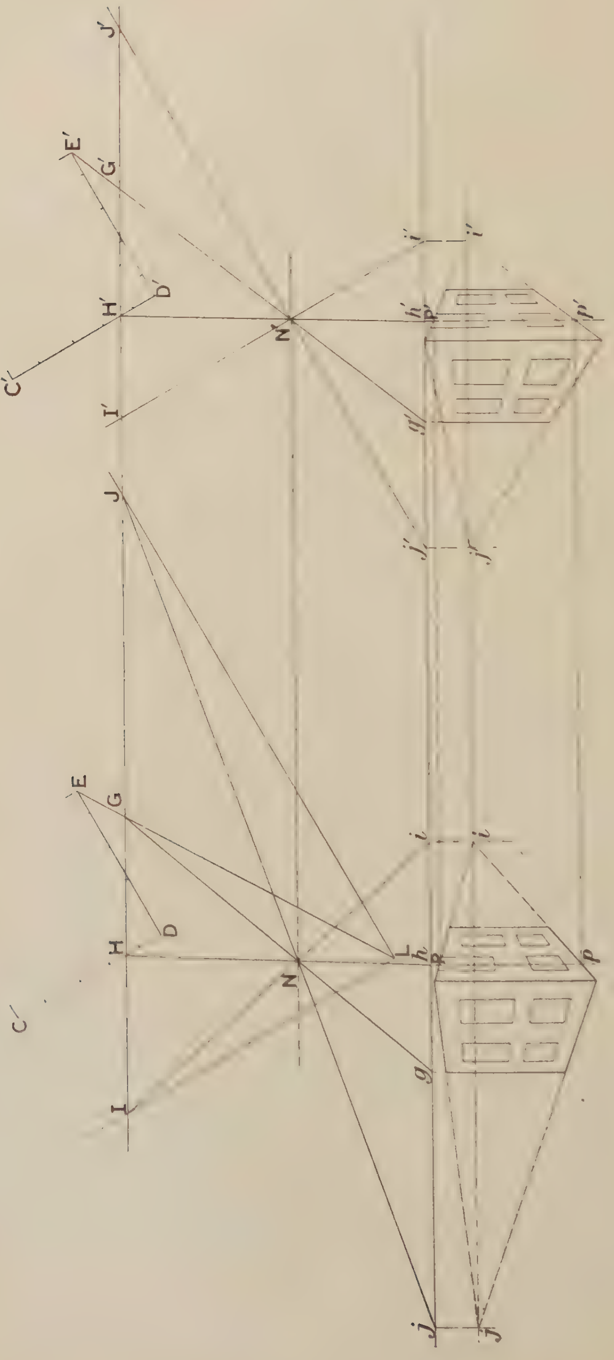

FIG. B.

FIG. A.

consider only the principal ray of each beam of rays from any point of the object. All these principal rays will be directed towards L. So far as the lens is concerned and the image which it produces on the plate, each point E of the building may then be replaced by the point G, in which EL meets the plane focussed on. The photograph is a copy on a definite scale of the projection thus obtained on this plane, and may be determined by joining each point G to N and (taking in the usual way for graphical construction the two nodal points in coincidence) producing this straight line on to meet the image plane in g. The vanishing points I and J for horizontal lines in the directions CD and DE are obviously found by drawing LI and LJ parallel to these directions, meeting the focussed plane in I and J. The images i and j of these points determined by the usual construction are the vanishing points for the photograph.

In Fig. A the point L is shown much nearer to the plate than the nodal point N, and in Fig. B, where corresponding accented letters are used, the case is shown where L′ is in coincidence with the nodal point. In both diagrams the focal length of the lens is the same, and the positions of the principal planes are identical. The two pictures differ very much from one another in their perspective and in the values of the lengths obtained by the construction suggested. The only parts of the two pictures in which the dimensions are equal are those such as Pp, P′p', in which the plane focussed for intersects the building. It is clear from these diagrams that the perspective, with a perfect lens such as is always assumed in considering elementary laws, depends on the position of the stop alone. If a lens of a different focal length were used to depict the same object, with its effective stop in the same position L, the perspective of the photograph would be exactly like that of Fig. A, though the two pictures would differ in size. Fig. B is the kind of case which Prof. Bryan has assumed to be general. This assumption is correct when the image is formed by a thin lens whose boundary is the limiting aperture, or when the lens is composed of two equal and similar components, with a stop placed symmetrically midway between them ; but many lenses are by no means like this. As extreme forms we have telephoto combinations, which are simply systems in which the nodal point N is situated a considerable distance in front of L, and at the same time the lenses are kept of reasonable dimensions by constructing them of a converging system situated in the neighbourhood of L, followed by a diverging system between the converging system and the plate.

The foregoing considerations make it clear that Prof. Bryan's interesting construction cannot be relied on to give even a rough indication of the focal length of the lens by which a given photograph has been taken.

Prof. G. H. Bryan communicated the following note in reference to the points raised by the Author in his reply : Referring to Fig. A, let M be the point on HNh at which ij subtends a right angle in the case of a rectangular building (or in the more general case an angle equal to the angle between the faces of the building). Then Mh is the length F_1 which I take for the focal length of the lens, while, according to Mr. Smith's construction, the focal length F should be Nh. (Of course this is the focal length *as focussed on the object*, and differs from the true focal length by the distance the lens has been displaced from the infinity position.) Now, it will be seen from similar triangles that

$$F_1 : F = LH : NH.$$

If the camera is focussed on a building or other object (as implied in the words "suitable object" in my note) whose distance is large compared with the distance LN, F_1 will be very approximately equal to F. If the camera is focussed for parallel rays the construction will be exact, at any rate under the assumptions involved in Mr. Smith's arguments. When writing the note I never contemplated the possibility that the method would be applied in cases where the dimensions of the optical system were comparable with the

distance of the object to be photographed. Mr. Smith admits that the method would be correct if the stop were placed at the optical centre of the system. If this is not the case, distortion of the image *may* take place, and in such cases errors would undoubtedly occur. As regards the effect of unequal trimming, the error introduced will be small if the faces of the building make angles of about 45 deg. with the line of sight. But the object which I had in view in connection with this method was to ascertain how wide-angled a lens would be required to take in the whole of an architectural subject of which a photograph was available, and it is clear that, if the picture has been trimmed, the estimate will be more than sufficient. The accuracy of the method, of course, depends on the accuracy with which the infinity points can be constructed in the photograph, and consequently, the method is limited to lenses which are not too narrow angled. I consider that the method is, therefore, correctly described in my note as a "rough-and-ready" method which is convenient for purposes such as those sketched out, for which Mr. Smith's methods would be unsuitable. A photographer using the method with camera in his own possession would naturally select a "suitable object," as suggested in my note, and would avoid using a rising front or trimming his print. In such cases it is impossible that the errors could approach anywhere near Mr. Smith's estimate, unless the lens system possessed excessive distortion. As regards the use of the method in connection with photographs taken under unknown conditions by other people, the method will certainly indicate conditions under which a similar photograph may be taken, and even in these cases it is difficult to see how the errors could possibly be so great as Mr. Smith estimates.

Mr. W. J. HALL (communicated) said that he had tried Prof. Bryan's method and found with a Cooke lens (nominal 5·5″ focal length) values 5·56″, 5·90″, 5·63″, 5·49″ and 5·50″ for the focal length, the distance between the lens and object being varied. With another lens of a different type, of which the focal length as determined by the very accurate method of Mr. Smith, was 14·60 cm., he obtained by Prof. Bryan's method 14·43 cm., 14·29 cm. and 14·30 cm. at different trials.

XII. *The Polyscope and its Projection. By* PROF. A. W. BICKERTON, *A.R.S.M.*

SHOWN AT THE MEETING ON JANUARY 22, 1915.

THE instrument consists of three narrow strips of plate glass about a foot long, arranged as a truncated pyramid. The object end has an area of about a quarter of a square inch, while the eye end is made very small (about a twentieth square inch) in order to get the maximum number of reflections. To get geometrical accuracy in the inter-facial angles the edges of the strips overlap instead of being butted. Three types were shown. In No. 1 the cross-section is an equilateral triangle. This produces the simplest type of pattern. In No. 2 the section is a right-angled isosceles triangle. This produces patterns with two centres of symmetry of eight reflections each and one of four reflections. In the third type the angles are 30 deg., 60 deg. and 90 deg. This produces patterns in which one centre has 12 reflections, one six and one four. When a group of nondescript objects are viewed through the instrument, an exquisitely coloured symmetrical pattern may be seen the character of which changes continuously as the object is moved about.

Suitable slides can readily be constructed by mounting pieces of coloured gelatine, fragments of lace, bits of wire, &c., on glass.

The designs were shown projected on a screen, a narrow beam of light from an arc lamp being passed through the polyscope, which was moved about until the maximum illumination was obtained. The beam before reaching the object slide passed through a water tank and a ground glass plate to ensure uniform brightness.

The instrument forms an invaluable aid to designers of tiles, floorcloths, fabrics, &c., as an innumerable sequence of designs can be reviewed by the artist in a short time, those which give satisfaction being sketched or photographed. If the slide is suitably mounted and moved reasonably slowly it is quite easy to repeat any particular pattern.

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